Resetting the Innovation Clock: Endogenous Growth through Technological Turnover*

Philippe Aghion[1] Antonin Bergeaud[2]

Timo Boppart[3] Jean-Félix Brouillette[4]

March 20, 2025

Abstract

We develop a growth model where even though ideas eventually become harder to find on any particular product line and long-run economic growth is constant despite sustained population growth, long-run innovation and growth are responsive to changes in market incentives, in particular to changes in market size. In the spirit of Aghion and Howitt (1996), entrants introduce new technologies through research and incumbents incrementally improve those technologies through development. Over time, however, it becomes harder to improve upon an existing technology such that incumbent firms eventually run out of ideas and exit the market. Their departure paves the way for new entrants that discover new technologies, thereby "resetting the innovation clock." It is this innovation reset process that generates sustained long-run endogenous economic growth. Analysis of the model reveals that, in a stationary equilibrium, the rate of economic growth is constant and endogenous despite a growing population and declining innovation efficiency at the firm-level. We present macro- and microlevel evidence that supports the predictions of our model.

*We are grateful to Conor Walsh for helpful comments.

[1]Collège de France, INSEAD, LSE, and CEPR. E-mail: philippe.aghion@insead.edu

[2]HEC Paris, CEP-LSE, and CEPR. E-mail: bergeaud@hec.fr

[3]IIES, Stockholm University and University of Zurich. E-mail: timo.boppart@iies.su.se

[4]HEC Montréal. E-mail: jean-felix.brouillette@hec.ca

1 Introduction

In this paper, we develop an endogenous growth model which captures three empirical facts which the existing theoretical literature typically fails to reconcile. First, despite sustained population growth and an expanding supply of researchers, economic growth has not accelerated in recent decades. Second, innovation is responsive to changes in market incentives, in particular to changes in market size, taxes, and R&D policy. Third, there are decreasing returns to R&D in any particular product line and ideas eventually become harder to find.

That economic growth remained steady despite population growth was first noted by Jones (1995). Examining data from 1953 to 1995, Jones (1995) finds that the nearly ninefold increase in R&D personnel did not translate into faster economic growth.

On the effects of changes in market size, taxes, and R&D policy on long-run innovation-led growth: Acemoglu and Linn (2004) document that, in the pharmaceutical industry, innovators tend to focus on drugs for wealthier consumers; similarly, recent work using French microdata by Aghion, Bergeaud, Lequien and Melitz (2024) finds that expanding a firm's export markets increases its output of pioneering patents. Akcigit, Grigsby and Nicholas (2017) provides historical evidence showing a positive causal effect of public investments. And Akcigit and Stantcheva (2020) document a negative effect of corporate income tax on long-term innovation and productivity growth.

Finally, on ideas becoming harder in particular product lines: Bloom, Jones, Van Reenen and Webb (2020) document this phenomenon in sectors such as semiconductors, agriculture, and health. For instance, while Moore's law suggests that the number of transistors on a chip should double roughly every two years, the number of researchers required to maintain this pace has increased dramatically—by a factor of eighteen since the early 1970s. A similar trend is observed in pharmaceuticals, where the number of new compounds approved by the FDA relative to research effort has fallen even as R&D employment has grown.

To account for the above empirical facts, in this paper we develop an endogenous growth model which features declining innovation efficiency at the product line level but yet sustained endogenous growth at the macro level. In the spirit of Aghion and Howitt (1996), what sustains economic growth in our model is the endogenous—research-driven—arrival of technological breakthroughs (i.e., new products, firms, or industries) that "reset" the research productivity clock continuously over time. Meanwhile, incumbent firms incrementally improve the quality of their products by investing in development. Over time, however, it becomes increasingly harder to improve product quality, to such an extent that incumbent firms eventually run out of ideas and exit the market. This in turn paves the way for new firms entering the market with new product lines where development-led quality improvements subsequently occur. This "resetting of the innovation clock" and the resulting turnover between exiting incumbent firms and the arrival of new firms with new product lines, gives rise to a continuous stream of temporary opportunities for product quality improvements that sustain economic growth.

In a stationary equilibrium, this turnover ensures that the total number of firms grows in proportion to the population, such that any expansion in the latter is absorbed by the proliferation of new firms. At the same time, our model displays declining innovation efficiency at the micro level which takes the form of an idiosyncratic "obsolescence" shock. Whenever a firm experiences this shock, its capacity to further improve its product's quality vanishes so that the firm's innovation efficiency instantly falls to zero.

Overall, economic growth is sustained by the continuous replacement of obsolete production lines of firms by new entrants, each bringing new opportunities for quality improvement. Importantly, at any point in time, a fraction of firms has not yet experienced the obsolescence shock, and their incentives to improve their product's quality remain responsive to policy. In fact, we show that, in stationary equilibrium, the growth rate of per capita consumption remains constant despite population growth, and yet it responds to changes in market size or innovation policy.

Literature Review

Our analysis in this paper first speaks to the long-standing economic history debate between those, who, following Gordon (2017), maintain that the era of great innovations is over and that innovation has entered a phase of sharply decreasing returns, and those who, following Joel Mokyr (e.g., see Mokyr, 2014), argue that the recent technological revolutions (IT and more recently AI), combined with the globalization of trade, have created the conditions for innovation and growth to prosper more than ever before. We contribute to this debate by developing a new growth model where ideas become harder to find within each particular line, but not in the economy as a whole as new lines keep being created.

Secondly, our paper can be seen as an attempt to settle the debate between endogenous growth and semi-endogenous growth advocates. On the one hand, a "first-generation" of innovation-based endogenous growth models (e.g., see Romer (1990); Aghion and Howitt (1992); and Aghion, Akcigit and Howitt (2015)) predict that market incentives—in particular property rights protection, market size, product market competition, or R&D subsidies—affect the long-run growth rate of the economy. However, in these models, population size had to remain constant in order for growth not to explode, a prediction which fails to withstand empirical scrutiny as many economies experienced roughly constant economic growth despite a growing research workforce. We depart from this first wave of endogenous growth models by developing a framework which delivers similar effects of market incentives despite having declining innovation efficiency at the product line level and without having to assume population growth away.

In contrast to the first-generation endogenous growth theories, Jones (1995)'s semi-endogenous growth model delivers constant long-run growth despite population growth. This in turn is achieved by allowing research productivity to decline with the accumulated stock of knowledge. A downside of that model, however, is that the long-run growth depends exclusively upon the rate of population growth (and a technology parameter), and in particular it is unaffected by market incentives or policy. We depart from the semi-endogenous growth paradigm by proposing a framework that generates a balanced growth path despite population growth, and yet where market incentives matter for long-run growth.

Most closely related to our analysis is the so-called "second generation" endogenous growth models (e.g., see Dinopoulos and Thompson (1998); Peretto (1998); Young (1998); and Howitt (1999)), which also deliver a constant long-run growth rate that can be affected by policy despite population growth. This in turn was achieved by allowing the number of firms to grow in proportion to the population (through free entry), keeping the ratio of researchers per firm constant so that each firm keeps growing at a constant rate. However, the assumption that research productivity should remain constant at the firm level, is directly questioned by the evidence in Bloom et al. (2020) showing that research productivity is instead declining at the firm–or product line–level. We depart from this second wave of endogenous growth models by developing a model which produced the same basic policy predictions as those delivered by the first-generation endogenous growth models, even though we allow for both, population growth and declining research productivity at the product line level. What sustains economic growth in our model is the endogenous arrival of technological breakthroughs (i.e., new products, firms, or industries) that "reset" the research productivity clock continuously over time.

The remaining part of the paper is organized as follows. Section 2 lays out our growth model. Section 3 solves the model. It computes the expression for the balanced growth rate, then presents a simple calibration of the model. Section 4 discusses the predictions of the model and compares them to empirical evidence. Section 5 analyzes the constrained-optimal allocation of the model. Section 6 concludes.

2 A New Growth Model

In this section, we present a simple model that captures the essence of our argument: firms invest in the *development* of existing lines to improve the quality of their products, but eventually run out of ideas and exit. Simultaneously, investments in *research* lead to the entry of new firms with entirely new product lines. As a consequence, opportunities for quality improvements are continuously replenished in the economy as a whole even though improvement opportunities are eventually exhausted on any existing lines.

2.1 The Economic Environment

Preferences

Consider a continuous-time economy where time is indexed by $t \in [0, \infty)$. This economy is populated by a representative household of measure N_t evolving according to:

$$\dot{N}_t = n \cdot N_t, \tag{1}$$

where $n > 0$. The household inelastically supplies one unit of labor at every point in time, and has logarithmic preferences over individual consumption c_t such that lifetime utility is defined as:

$$U_0 = \int_0^\infty e^{-(\rho-n)t} \ln(c_t) \mathrm{d}t. \tag{2}$$

Here, $\rho > n$ denotes the rate of time preference.

Technology

The economy is composed of a final sector producing a final good using a Dixit and Stiglitz (1977) aggregate of differentiated products indexed by $i \in \mathcal{I}_t$:

$$Y_t = \left(\int_{i \in \mathcal{I}_t} (q_{it} \cdot y_{it})^{\frac{\theta-1}{\theta}} \mathrm{d}i \right)^{\frac{\theta}{\theta-1}}, \tag{3}$$

where Y_t is the quantity of the final good produced at time t, y_{it} is the quantity of product i supplied at time t, $q_{it} > 0$ is the quality of that product, $\theta > 1$ is the elasticity of substitution between products, $\mathcal{I}_t$ is the set of available products, and its (endogenous) cardinality is denoted by $M_t \equiv |\mathcal{I}_t|$.

Here, a "product" is best understood as a new technology rather than an incremental improvement on an existing one. For instance, early film cameras included simple box cameras like the Kodak Brownie, which had fixed-focus lenses and limited control over exposure. Over time, manufacturers introduced more advanced 35 mm and single-lens reflex (SLR) film cameras–such as the Nikon F or the Canon AE-1–with features like interchangeable lenses, better metering systems, and more precise shutter controls. These advancements represented successive improvements of film camera technology. Eventually, however, those improvement opportunities in film-based imaging were depleted, paving the way for a fundamentally new product: the digital camera, which opened up new dimensions of innovation (e.g., image sensors, on-board processing, and software-driven features).

Each of those products is produced by a single firm using labor according to the linear technology:

$$y_{it} = l_{it}, \tag{4}$$

where y_{it} denotes the quantity of product i supplied at time t and l_{it} denotes the quantity of labor used in production.

Over time, a firm can incrementally improve the quality of its product by directing final goods towards development. More precisely, a product's quality evolves according to the following controlled process:[1]

$$\dot{q}_{it} = \gamma_{it} \cdot q_{it}, \tag{5}$$

where γ_{it} is the proportional drift of the product's quality. The final good requirement d_{it} to achieve this quality drift is given by:

$$d_{it} = Q_t N_t^{\frac{1}{\theta-1}} \cdot (q_{it}/Q_t)^{\theta-1} \cdot \frac{c_D \gamma_{it}^{1+\zeta}}{1+\zeta} \tag{6}$$

where $c_D > 0$ determines the scale of the development cost function, $\zeta > 0$ measures its elasticity, and Q_t is an average quality index defined as:[2]

$$Q_t \equiv \left(M_t^{-1} \int_{i \in \mathcal{I}_t} q_{it}^{\theta-1} \mathrm{d}i \right)^{\frac{1}{\theta-1}}.$$

The scaling of the development cost function by $Q_t N_t^{\frac{1}{\theta-1}}$ is designed to capture the effect that a larger market leads to faster economic growth. Alternatively, one could move

[1]Here, we specify a deterministic process for the sake of tractability, but the model could easily be extended to allow for idiosyncratic quality shocks.

[2]The development technology depends on a product's quality relative to the average index q_{it}/Q_t to ensure that product quality grows at the same rate for all firms in equilibrium (Gibrat's law), which delivers an analytical solution for the market allocation in this model.

away from a CES production function—as in Latzer, Matsuyama and Parenti (2019)—to disentangle the effect of a larger market from that of a larger population.

However, at Poisson rate $\epsilon > 0$, a firm may receive an idiosyncratic "obsolescence" shock after which it can no longer improve its product's quality. This is an extreme and stylized case of "ideas becoming harder to find" (Bloom et al., 2020) in which innovation efficiency literally falls to zero once the shock hits.[3] After being hit by this shock, the product quality eventually falls below a certain threshold $\underline{q}_t = \underline{q} \cdot Q_t$ where $\underline{q} \in (0, 1)$, the firm exogenously exits the market.[4]

In every point in time, a unit measure of potential entrants attempt to discover products that are entirely new to society through research. Specifically, these entrants can direct $c_R Q_t M_t^{\frac{1}{\theta-1}}$ units of final goods to research in order to invent a unit flow of these new products. Once a product is discovered, its initial quality is drawn from a point mass at the lower bound $\underline{q}_t$ of the product quality support.[5]

Resource Constraints

Labor supplied by the household can be allocated to either production, delivering the labor market clearing condition:

$$L_t \equiv \int_{i \in \mathcal{I}_t} l_{it} \mathrm{d}i \leq N_t. \tag{7}$$

and the final good can be allocated to either consumption, research, or development, which delivers following resource constraint:

$$C_t + R_t + D_t \leq Y_t \quad \text{where} \quad D_t \equiv \int_{i \in \mathcal{I}_t} d_{it} \mathrm{d}i \quad \text{and} \quad R_t \equiv c_R Q_t M_t^{\frac{1}{\theta-1}} E_t. \tag{8}$$

Here, E_t denotes the aggregate flow of new products.

[3] One could instead consider a more general formulation by which the efficiency of the development technology gradually decreases as a product's quality is improved.

[4] This simple exit rule is chosen for tractability as it delivers an analytic solution to the growth rate of consumption per capita, but it could easily be extended to an optimal stopping time problem in the presence of overhead costs.

[5] One could specify a more general learning process between entrants and incumbents (see Yao (2024) for a very general and still tractable specification), but we choose to keep the model as simple as possible to illustrate the driving mechanisms.

2.2 The Decision Problems

In this section, we define the decision problems of each economic agent. In terms of market structure, we assume that all agents take part in perfect competition in all markets besides firms who engage in monopolistic competition.

The Household's Problem

Taking prices as given, the household's problem is to choose its consumption to maximize lifetime utility:

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-(\rho-n)t} \ln(c_t) \mathrm{d}t$$

subject to the flow budget constraint:

$$\dot{a}_t + P_t c_t \leq (r_t - n)a_t + w_t - T_t.$$

Here, w_t is the wage rate, T_t are lump-sum per-capita taxes, a_t is the value of corporate assets per capita, and r_t is the rate of return on those assets:

$$a_t N_t = \int_{i \in \mathcal{I}_t} V_{it} \mathrm{d}i \quad \text{where} \quad \lim_{t \to \infty} e^{-\int_0^t (r_{t'} - n)\mathrm{d}t'} a_t = 0,$$

and where V_{it} denotes the value of product i at time t. This problem thus delivers the usual intertemporal Euler equation:

$$\dot{c}_t = (r_t - \rho)c_t.$$

The Final Sector's Problem

Taking prices as given, the perfectly competitive final sector's problem is to choose its demand for each product to maximize profits:

$$\max_{\{\{c_{it}\}_{i \in \mathcal{I}_t}\}_{t \geq 0}} \{P_t Y_t - \int_{i \in \mathcal{I}_t} p_{it} y_{it} \mathrm{d}i\} \quad \text{s.t.} \quad Y_t = \left(\int_{i \in \mathcal{I}_t} (q_{it} y_{it})^{\frac{\theta-1}{\theta}} \mathrm{d}i \right)^{\frac{\theta}{\theta-1}}$$

This problem thus delivers the following demand functions:

$$y_{it} = (P_t / p_{it})^\theta q_{it}^{\theta-1} Y_t.$$

Since aggregate consumption is chosen as the numéraire, the price index P_t is normalized to one for all t:

$$P_t \equiv \left(\int_{i \in \mathcal{I}_t} (p_{it}/q_{it})^{1-\theta} \mathrm{d}i \right)^{\frac{1}{1-\theta}} = 1.$$

The Firms' Problem

Firms engage in monopolistic competition in the product market but perfect competition in the labor market, taking the wage and the demand for their product as given. A firm thus chooses the price at which to sell its product and its final goods and labor demands to maximize the expected present discounted value of its profits.

From this point on, we abandon the i-index notation since a firm is entirely described by its product's quality q and its obsolescence status. We denote the latter by $S \in \{O, N\}$ where O represents an "old" firm that has received the obsolescence shock, and N represents a "new" firm that has not. The new firm's value function satisfies a standard Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} r_t V_t^N(q) = \max_{\mathbf{u}_t^N(q) \geq \mathbf{0}} \{ &(1-\tau^C)[p_t(q)y_t(q) - w_t l_t(q)] + (1-\tau^D)d_t(q) \\ &+ \gamma_t(q) q \partial_q V_t^N(q) \} + \epsilon [V_t^O(q) - V_t^N(q)] + \dot{V}_t^N(q) \end{aligned}$$

where $\mathbf{u}_t^N(q) \equiv \{p_t(q), l_t(q), d_t(q)\}$ is the vector of control variables, $\tau^C > 0$ is the corporate income tax rate, and $\tau^D > 0$ is a subsidy on the firm's development expenditures. Similarly, the "old" firm's value function satisfies the HJB equation:

$$r_t V_t^O(q) = \max_{\mathbf{u}_t^O(q) \geq \mathbf{0}} \{ (1-\tau^C)[p_t(q)y_t(q) - w_t l_t(q)] \} + \dot{V}_t^O(q)$$

where $\mathbf{u}_t^O(q) \equiv \{p_t(q), l_t(q)\}$. The profit-maximization problem implies that a firm sets its price to a constant markup above marginal cost irrespective of its obsolescence status:

$$p_t(q) = \mu \cdot w_t, \ \forall q \quad \text{where} \quad \mu \equiv \frac{\theta}{\theta - 1}.$$

Hence, a firm's flow profits can be expressed as:

$$\pi_t^S(q) = (1-\tau^C)(q/Q_t)^{\theta-1} Y_t / (\theta M_t) - \mathbb{1}_{\{S=N\}} (1-\tau^D) d_t(q),$$

where the optimal labor allocation to development is given by:

$$d_t(q) = \frac{c_D Q_t^\theta N_t^{\frac{1}{\theta-1}} \gamma_t(q)^{1+\zeta}}{(1+\zeta)q^{\theta-1}},$$

and the resulting quality drift is:

$$\gamma_t(q) = \left[\frac{q\partial_q V_t^N(q)}{(1-\tau^D)Q_t N_t^{\frac{1}{\theta-1}} c_D (q/Q_t)^{\theta-1}}\right]^{1/\zeta}.$$

The Entrant's Problem

Entrants engage in perfect competition on the final goods market and, thus, choose their allocation R_t of final goods to research to maximize future expected profits while taking the wage rate as given:

$$V_t^E = \max_{R_t} \left\{ \frac{V_t^N(\underline{q}_t)R_t}{c_R Q_t M_t^{\frac{1}{\theta-1}}} - (1-\tau^R)R_t \right\}$$

where $\tau^R > 0$ is a research subsidy. The first-order condition of the entrant's problem delivers what will be referred to as the free-entry condition:

$$V_t^N(\underline{q}_t) = (1-\tau^R)c_R Q_t M_t^{\frac{1}{\theta-1}}.$$

2.3 The Equilibrium Market Allocation

Having defined the decision problems of each economic agent, we can now define the concept of an equilibrium market allocation, and lay out the equations that determine the long-run equilibrium growth rate of the aggregate economy.

Definition 1. *Given the initial conditions $\{N_0, Q_0, \{m_0^N(q), m_0^O(q)_{q=\underline{q}_t}^\infty\}\}$, where $m_0^T(q)$ is the initial measure of type-T firms with product quality q, a market allocation consists of time paths for quantities, prices, and policy functions such that the following conditions hold:*

1. $\{c_t\}_{t\geq 0}$ *solve the household's problem.*
2. $\{\{p_t(q), l_t(q), d_t(q)\}_{q=\underline{q}_t}^\infty\}_{t\geq 0}$ *solve the firm's problem.*
3. $\{R_t\}_{t\geq 0}$ *solve the entrant's problem.*

4. *$\{\{p_t(q)\}_{q=\underline{q}_t}^{\infty}\}_{t\geq 0}$ clear the product markets.*

5. *$\{w_t\}_{t\geq 0}$ clear the labor market.*

6. *$\{r_t\}_{t\geq 0}$ clear the asset market.*

7. *The government's budget is balanced.*

The market allocation in this model admits a remarkably simple aggregation such that output per capita is given by:

$$y_t = M_t^{\frac{1}{\theta-1}} Q_t.$$

That is, consumption per capita is increasing in the measure of products (owing to a taste for variety) and the average quality across those products.

3 Solving the Model

3.1 Equilibrium Balanced Growth Path

In Appendix A.1 we show that on a balanced growth path (BGP), the measure of products grows at the same rate as the population, the average quality index grows at a constant rate, and, thus, the growth rate of consumption per capita is constant and given by:

$$g = \frac{n}{\theta - 1} + g^Q. \tag{9}$$

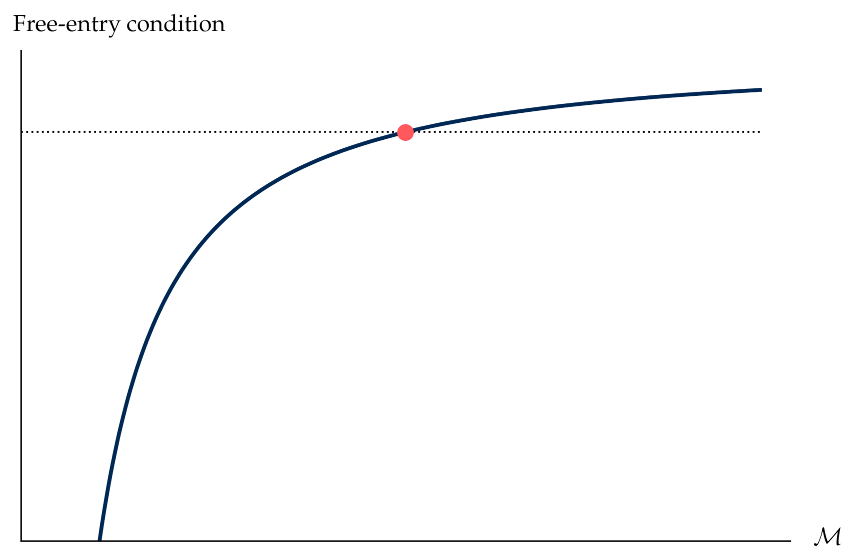
We can derive an expression for the growth rate of the average quality index:

$$g^Q = \gamma \cdot \frac{n+d}{n+\epsilon} - \frac{n(1-\underline{q}^{\theta-1})}{\theta-1} \quad \text{where} \quad \gamma = \left[\frac{(\theta-1)(1-\tau^R)c_R \mathcal{M}^{\frac{1}{\theta-1}}}{(1-\tau^D)c_D \underline{q}^{\theta-1}} \right]^{1/\zeta} \tag{10}$$

which is composed of two additive terms. The first term reflects the positive growth contribution of incremental product quality improvements by incumbent firms. This term is the product of the common product quality drift γ among firms that haven't yet received the obsolescence shock and the fraction of such firms in the economy given by $\frac{n+d}{n+\epsilon}$. The second term reflects the negative growth contribution of net product entry occurring at rate n. Indeed, entering firms draw their initial product quality at the lower bound $\underline{q} \in (0,1)$ of the quality support, which drags down the average quality index.

Notice that the quality drift depends on the measure of products per capita (denoted by $\mathcal{M}_t \equiv M_t/N_t$), which is determined in equilibrium through the free-entry condition illustrated in Figure 1 using the parameter values detailed in Section 3.2. In Appendix A, we show that this equilibrium is unique.

Figure 1: The Market Allocation



Note: The root of this equation determines the unique value of $\mathcal{M}$ that satisfies the free-entry condition.

3.2 A Simple Calibration

In this section, we present a straightforward calibration of the parameters of our model to illustrate its mechanisms. Although this exercise is not meant as a rigorous empirical quantification for direct comparison with economic data, it provides insight into the potential magnitude of different forces.

We set the pure rate of time preference, ρ, to 0.04 and assume an annual population growth rate, n, of 1%. Consistent with Garcia-Macia, Hsieh and Klenow (2019), the elasticity of substitution across products, θ, is set to 4. We fix ζ, which captures the degree of decreasing returns to development labor, at 1—aligning with microeconomic evidence on the tax elasticity of R&D summarized in Acemoglu, Akcigit, Alp, Bloom and Kerr (2018).

Four additional parameters require calibration: the development cost parameter c_D, the research cost parameter c_R, the initial quality of new entrants $\underline{q}$, and the obsolescence shock ϵ. Since these parameters are less conventional in the literature or not directly observable, we calibrate them by jointly matching the following four empirical moments:

1. Per capita consumption grows at 2% per year.
2. The firm-level entry rate is 10%.
3. The average firm has 20 employees.
4. New entrants are 95% as large as incumbents.

The first three moments are representative of the U.S. economy, while the fourth reflects a stylized fact from Einav, Klenow, Murciano-Goroff and Levin (2022), who document that the average entrant achieves roughly the same sales per customer as the average incumbent. The calibrated parameter values are reported in Table 1.

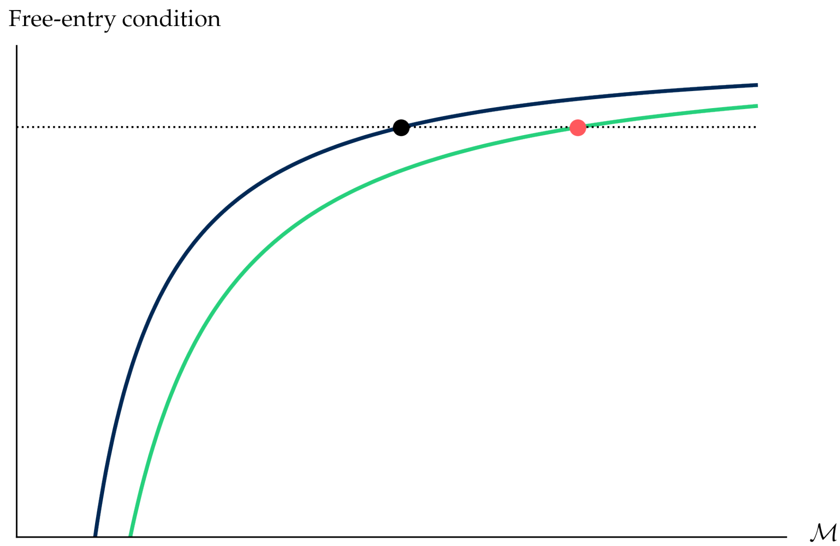
Table 1: Calibration

Parameter	Value	Source
ρ	0.04	Standard
n	0.01	Population growth
θ	4	Garcia-Macia et al. (2019)
ζ	1	Acemoglu et al. (2018)
c_D	$\exp(8.48)$	Consumption per capita growth
c_R	$\exp(4.34)$	Average firm size
$\underline{q}$	0.1	Relative size of entrants
ϵ	0.1	Entry rate

While the parameters $\{c_D, c_R, \underline{q}, \epsilon\}$ are jointly calibrated to match the aforementioned moments, we provide some intuition for their identification. The development cost parameter c_D is primarily identified by the growth rate of per capita consumption, as it governs the pace at which firms improve their product's quality and, in turn, economic growth. The research cost parameter c_R is determined by the average firm size, since it influences the cost of entry and thereby the equilibrium number of firms. The initial quality of new entrants $\underline{q}$ is directly linked to the relative size of entrants compared to incumbents. Finally, the obsolescence shock, ϵ, is identified from the firm-level entry rate because it governs how quickly firms become obsolete and ultimately exit the market. Under a balanced growth path, the entry rate equals the sum of the exit rate and the population growth rate.

Under this calibration, Figure 2 illustrates the equilibrium effect of introducing a 100% corporate income subsidy–equivalent to doubling the size of the market. This subsidy shifts the free-entry condition to the right, resulting in a higher equilibrium measure of products per capita and, therefore, increasing the growth rate of consumption per capita by 45 basis points.

Figure 2: Corporate Income Subsidy (τ^C)

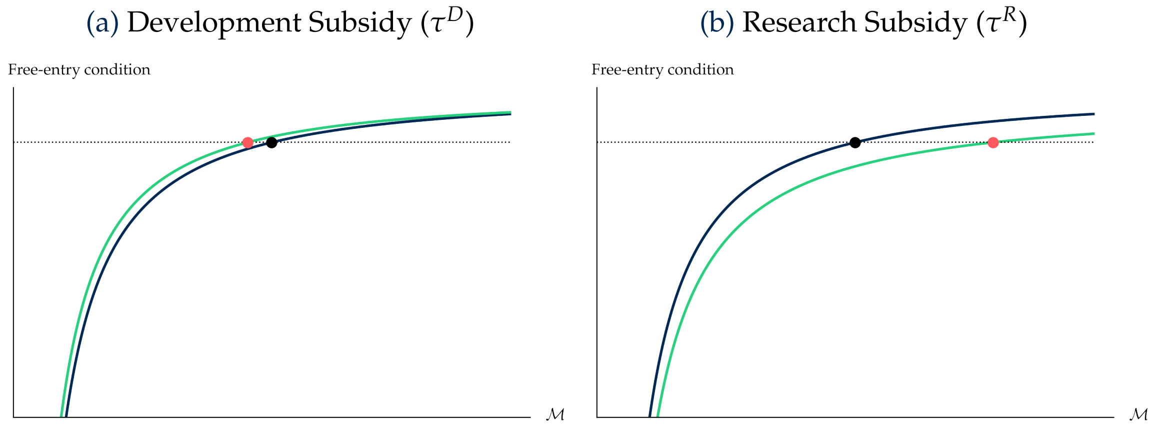


Note: The green line represents the free-entry condition when introducing a corporate income subsidy of 100%.

Figure 3 illustrates the equilibrium effects of R&D policy. In Panel 3(a), a 30% subsidy on incumbent firms' product development expenditures is implemented. Although this subsidy raises the per capita consumption growth rate by 71 basis points, it has little to no effect on the equilibrium value of $\mathcal{M}$. This indicates that the subsidy primarily affects economic growth through its direct effect on the product quality drift γ, rather than through the measure of products per capita.

In contrast, Panel 3(b) illustrates that a 30% subsidy on research expenditures for new entrants substantially increases the equilibrium value of $\mathcal{M}$. This policy has two opposing effects on economic growth. On one hand, the lower entry cost reduces the firm's value through the free-entry condition, thereby dampening development incentives. On the other hand, the higher measure of products per capita increases the cost of entry relative to development. Ultimately, the first effect dominates, resulting in a 36 basis point decline in the growth rate of consumption per capita.

Figure 3: The Equilibrium Effects of R&D Policy



Note: The green lines represent the free-entry condition when introducing subsidies of 30% to development and research, respectively.

4 Discussion

While partly building on it and sharing an important empirical motivation with it, our model departs from the canonical semi-endogenous growth framework in ways which allow us to match additional empirical facts.

First, our model predicts that market size matters for long-run innovation and growth. In the growth formula (10), the market size effect is captured by the term $\mathcal{M}^{\frac{1}{\theta-1}}$. In words, as the market expands—and therefore profits increase—more firms are incentivized to enter, thereby raising the number of firms per capita. A higher firm density in turn reduces the relative cost of development compared to research, ultimately accelerating TFP growth. This market size prediction is borne by several empirical studies show that larger market size, e.g., driven by an increased market demand, stimulates innovation and growth in the long run. Thus, at the sectoral level, Acemoglu and Linn (2004) find that a 1% expansion in a U.S. pharmaceutical market leads to a 4 to 6% rise in new drugs. Dubois, De Mouzon, Scott-Morton and Seabright (2015), using updated data, report a smaller elasticity of 0.23, implying that a 10% market increase induces a 2.3% rise in drug development, with \$2.5 billion in additional revenue needed per new molecule. Blume-Kohout and Sood (2013) show that Medicare Part D in 2006 significantly boosted pharmaceutical revenues and led to a 50% increase in new clinical trials. In vaccines, Finkelstein (2004) documents that policy-driven market expansions increased vaccine trials by 2.5 times, with \$0.06 of R&D investment per additional dollar of anticipated revenue. At the firm level, Aghion et al. (2024) study French manufacturers and find that the most productive firms increase patenting in response to export demand shocks. They interpret this fact using the lens of the Schumpeterian growth model, where enhanced competition discourages less efficient firms from innovating with a counteracting effect driven by the positive effect of competition on market size and therefore on innovation. Similarly, Bustos (2011) finds that Argentinian firms exposed to MERCOSUR tariff reductions in the 1990s adopted advanced technologies and intensified R&D and Lileeva and Trefler (2010) show that Canadian manufacturers entering export markets after the Canada—US Free Trade Agreement experienced productivity gains, product innovation, and technological upgrades.

Second, our model predicts that tax policy impacts long term innovation and growth. For example, the growth formula in equation (10) implies that an increase in τ^D has a negative effect on long-run innovation and growth. Empirically we know that R&D tax credits impact long-run innovation (e.g., see (Dechezleprêtre, Einiö, Martin, Nguyen and Van Reenen, 2023)) even when these are not necessarily the most efficient instruments to stimulate certain kinds of innovation. Moreover, the same growth formula shows that an increase in corporate taxation τ^R leads to a decline in growth and innovation, in line with Kennedy, Dobridge, Landefeld and Mortenson (2022); Mukherjee, Singh and Žaldokas (2017).

A third departure from the semi-endogenous growth model, is that in our model innovation by incumbent firms is an important driver of growth. In fact, Garcia-Macia et al. (2019) document that over half of U.S. TFP growth between 1983 and 2013 is attributable to existing product improvements by incumbent firms.

We can modify our model in two ways to recover the canonical semi-endogenous growth framework. First, we can rule out quality improvements of existing products (i.e., take $c_D \to \infty$). Under this assumption, all products retain a constant quality level, so growth is driven solely by the entry of new products. In this case, the per capita consumption growth rate becomes:

$$g = \frac{n}{\theta - 1}$$

which is the standard result in the semi-endogenous growth literature. However, this modification fails to capture the empirical observation that improvements in existing products are the primary driver of growth in the U.S. economy.

Second, we could assume that new entrants draw their initial product quality from a fixed distribution that does not scale with the existing average quality index. With this assumption, new products begin at a given quality level, improve until the obsolescence shock occurs, and then stagnate, modification of the model, however, also contradicts the empirical findings of Garcia-Macia et al. (2019), since it implies that product improvements play no role in driving overall economic growth.

5 The constrained-optimal Allocation

In Appendix A.2, we pose the problem of a planner that maximizes social welfare while taking the incumbent-to-incumbent and incumbent-to-entrant technology spillovers as given. More precisely, the planner takes Q_t and M_t in the research and development technologies as given. Because the planner does not internalize such spillovers, we refer to the solution to this problem as *constrained*-optimal.

To decentralize this constrained-optimal allocation, we show that a government could implement subsidies to both research and development of $\tau^R = \tau^D = 1/\theta$. That is, the market allocation features too little research (entry) and development, disregarding technology spillovers. As in Dhingra and Morrow (2019), there is too little entry because new entrants do not internalize that they raise demand for existing firms due to the "taste for variety." Moreover, there is too little development because firms exert market power. Indeed, firms produce at levels below what would prevail under perfect competition, which reduces the "market size" incentive for development. Note, however, that if our model featured head-to-head creative destruction from research, the market allocation could feature too much of it.

6 Conclusion

In this paper, we develop a growth model where, even though ideas eventually become harder to find on any particular product line and long-run economic growth remains constant despite assuming population growth, long-run innovation and growth are responsive to changes in market incentives, in particular to changes in market size. In other words, starting from the same premises as the semi-endogenous growth model, we end up with a fully-fledged endogenous growth model. What leads to long-run endogenous economic growth is that whenever existing firms run out of steam in improving quality on their existing lines, new entrants keep "resetting the innovation clock" by introducing new technologies, i.e., new lines that are improved upon until subsequent lines are created. This continuous reset process moves us back from semi-endogenous growth to endogenous growth.

Our approach in this paper can be extended in several directions. One extension would be to allow incumbent firms on any existing line to either pursue development activities on this line or engage again in research to create new lines, thereby introducing directed technical change as in Aghion and Howitt (1996). A second extension would be to allow for creative destruction within and across lines, and more generally to embed the model in this paper into a fully-fledged Schumpeterian growth model. As pointed out in the previous section, allowing for creative destruction would affect the comparison between the laissez-faire equilibrium and the constrained social optimum. A third extension would be to explore the innovation reset mechanism in particular sectors or research areas. For example, to what extent can we say that ideas eventually become harder to find in any particular subfield of economics and yet overall research productivity in economics never wavers out as new subfields keep being discovered? This and other extensions of our analysis in this paper are left for future research.

References

Acemoglu, Daron and Joshua Linn, "Market Size in Innovation: Theory and Evidence from the Pharmaceutical Industry," *The Quarterly Journal of Economics*, 08 2004, *119* (3), 1049–1090.

_ , Ufuk Akcigit, Harun Alp, Nicholas Bloom, and William Kerr, "Innovation, Reallocation, and Growth," *American Economic Review*, November 2018, *108* (11), 3450–91.

Aghion, Philippe and Peter Howitt, "A Model of Growth Through Creative Destruction," *Econometrica*, 1992, *60* (2), 323–351.

_ and _ , "Research and Development in the Growth Process," *Journal of Economic Growth*, 1996, *1*, 49–73.

_ , Antonin Bergeaud, Matthieu Lequien, and Marc J. Melitz, "The Heterogeneous Impact of Market Size on Innovation: Evidence from French Firm-Level Exports," *The Review of Economics and Statistics*, 05 2024, *106* (3), 608–626.

_ , Ufuk Akcigit, and Peter Howitt, "Lessons from Schumpeterian Growth Theory," *American Economic Review*, May 2015, *105* (5), 94–99.

Akcigit, Ufuk and Stefanie Stantcheva, *Taxation and Innovation: What Do We Know?*, University of Chicago Press, November 2020.

_ , John Grigsby, and Tom Nicholas, "Immigration and the Rise of American Ingenuity," *American Economic Review*, May 2017, *107* (5), 327–31.

Bloom, Nicholas, Charles I. Jones, John Van Reenen, and Michael Webb, "Are Ideas Getting Harder to Find?," *American Economic Review*, April 2020, *110* (4), 1104–44.

Blume-Kohout, Margaret E and Neeraj Sood, "Market size and innovation: Effects of Medicare Part D on pharmaceutical research and development," *Journal of public economics*, 2013, *97*, 327–336.

Bustos, Paula, "Trade liberalization, exports, and technology upgrading: Evidence on the impact of MERCOSUR on Argentinian firms," *American economic review*, 2011, *101* (1), 304–340.

Dechezleprêtre, Antoine, Elias Einiö, Ralf Martin, Kieu-Trang Nguyen, and John Van Reenen, "Do tax incentives increase firm innovation? an rd design for r&d, patents, and spillovers," *American Economic Journal: Economic Policy*, 2023, *15* (4), 486–521.

Dhingra, Swati and John Morrow, "Monopolistic Competition and Optimum Product Diversity under Firm Heterogeneity," *Journal of Political Economy*, 2019, *127* (1), 196–232.

Dinopoulos, Elias and Peter Thompson, "Schumpeterian Growth Without Scale Effects," *Journal of Economic Growth*, 1998, *3* (4), 313–335.

Dixit, Avinash K. and Joseph E. Stiglitz, "Monopolistic Competition and Optimum Product Diversity," *The American Economic Review*, 1977, *67* (3), 297–308.

Dubois, Pierre, Olivier De Mouzon, Fiona Scott-Morton, and Paul Seabright, "Market size and pharmaceutical innovation," *The RAND Journal of Economics*, 2015, *46* (4), 844–871.

Einav, Liran, Peter J. Klenow, Raviv Murciano-Goroff, and Jonathan D. Levin, "Customers and Retail Growth," Technical Report, Stanford University Manuscript 2022.

Finkelstein, Amy, "Static and dynamic effects of health policy: Evidence from the vaccine industry," *The Quarterly Journal of Economics*, 2004, *119* (2), 527–564.

Garcia-Macia, Daniel, Chang-Tai Hsieh, and Peter J. Klenow, "How Destructive Is Innovation?," *Econometrica*, 2019, *87* (5), 1507–1541.

Gordon, Robert, *The Rise and Fall of American Growth: The U.S. Standard of Living Since the Civil War*, Princeton University Press, 2017.

Howitt, Peter, "Steady Endogenous Growth with Population and R & D Inputs Growing," *Journal of Political Economy*, 1999, *107* (4), 715–730.

Jones, Charles I., "R & D-Based Models of Economic Growth," *Journal of Political Economy*, 1995, *103* (4), 759–784.

Kennedy, Patrick J, Christine Dobridge, Paul Landefeld, and Jake Mortenson, "The efficiency-equity tradeoff of the corporate income tax: Evidence from the tax cuts and jobs act," 2022. Working Paper.

Latzer, Hélène, Kiminori Matsuyama, and Mathieu Parenti, "Reconsidering the Market Size Effect in Innovation and Growth," Technical Report, CEPR Discussion Paper No. 14250 2019.

Lileeva, Alla and Daniel Trefler, "Improved access to foreign markets raises plant-level productivity… for some plants," *The Quarterly journal of economics*, 2010, *125* (3), 1051–1099.

Mokyr, Joel, "Secular stagnation? Not in your life," in Coen Teulings and Richard Baldwin, eds., *Secular Stagnation: Facts, Causes and Cures*, Paris & London: CEPR Press, 2014.

Mukherjee, Abhiroop, Manpreet Singh, and Alminas Žaldokas, "Do corporate taxes hinder innovation?," *Journal of Financial Economics*, 2017, *124* (1), 195–221.

Peretto, Pietro F., "Technological Change and Population Growth," *Journal of Economic Growth*, 1998, *3* (4), 283–311.

Romer, Paul M., "Endogenous Technological Change," *Journal of Political Economy*, 1990, *98* (5, Part 2), S71–S102.

Yao, Jinglun, "Knowledge is (Market) Power," Working Paper 2024.

Young, Alwyn, "Growth without Scale Effects," *Journal of Political Economy*, 1998, *106* (1), 41–63.

Appendix

A Theoretical Appendix

This section of the Appendix provides derivations and proofs for the results presented in the paper.

A.1 The Market Allocation

Hamilton-Jacobi-Bellman Equations. The old firm's value function satisfies the HJB equation:

$$r_t V_t^O(q) = (1-\tau^C)(q/Q_t)^{\theta-1} y_t N_t/(\theta M_t) + \dot{V}_t^O(q).$$

Defining $x_t \equiv \ln(q_t/Q_t)$, we can rewrite:

$$r_t V_t^O(x) = (1-\tau^C)\exp[(\theta-1)x] y_t N_t/(\theta M_t) - g_t^Q \partial_x V_t^O(x) + \dot{V}_t^O(x)$$

where $g_t^Q \equiv \dot{Q}_t/Q_t$ denotes the growth rate of the average quality index. Let us guess that this value function takes the following form:

$$V_t^O(x) = V_t^O \exp[(\theta-1)x].$$

Substituting this guess in the HJB equation, we obtain the following ordinary differential equation (ODE):

$$\dot{V}_t^O = [r_t + (\theta-1)g_t^Q]V_t^O - (1-\tau^C) y_t N_t/(\theta M_t),$$

which verifies our guess. The new firm's value function satisfies the HJB equation:

$$\begin{aligned}(r_t+\epsilon)V_t^N(q) = {} & (1-\tau^C)(q/Q_t)^{\theta-1} y_t N_t/(\theta M_t) - (1-\tau^D) d_t(q) \\ & + \gamma_t(q) q \partial_q V_t^N(q) + \epsilon V_t^O(q) + \dot{V}_t^N(q).\end{aligned}$$

Using the change of variable defined above, we can rewrite:

$$\begin{aligned}(r_t+\epsilon)V_t^N(x) = {} & (1-\tau^C)\exp[(\theta-1)x] y_t N_t/(\theta M_t) - (1-\tau^D) d_t(x) \\ & + [\gamma_t(x) - g_t^Q]\partial_x V_t^N(x) + \epsilon V_t^O(x) + \dot{V}_t^N(x).\end{aligned}$$

Let us guess that this value function takes the following form:

$$V_t^N(x) = V_t^N \exp[(\theta-1)x].$$

Substituting this guess in the optimal product quality drift, we obtain:

$$\gamma_t = \left[\frac{(\theta - 1)V_t^N}{(1 - \tau^D)c_D Q_t N_t^{\frac{1}{\theta-1}}} \right]^{1/\zeta}$$

which is independent of a product's quality. Substituting this result in the HJB equation of the new firm, we obtain the following ODE, which verifies our guess:

$$\dot{V}_t^N = [r_t + \epsilon - (\theta - 1)(\gamma_t - g_t^Q)]V_t^N - \epsilon V_t^O - \frac{(1 - \tau^C)y_t N_t}{\theta M_t} - \frac{(1 - \tau^D)c_D Q_t N_t^{\frac{1}{\theta-1}} \gamma_t^{1+\zeta}}{1 + \zeta}.$$

Fokker-Planck Equations. The Fokker-Planck (FP) equations describing the evolution of the density of log relative quality among new and old firms are given by:

$$\begin{aligned}
\dot{m}_t^N(x) &= -(\gamma_t - g_t^Q)\partial_x m_t^N(x) - \epsilon m_t^N(x) + \delta(x - \underline{x}_t)R_t, \\
\dot{m}_t^O(x) &= g_t^Q \partial_x m_t^O(x) + \epsilon m_t^N(x),
\end{aligned}$$

where $\delta(\cdot)$ denotes the Dirac delta function and $\underline{x}_t \equiv \ln(\underline{q}_t / Q_t)$. Therefore, the law of motion for the measure of new and old products are given by:

$$\dot{M}_t^N = R_t - \epsilon M_t^N \quad \text{and} \quad \dot{M}_t^O = \epsilon M_t^N - d_t M_t$$

where d_t denotes the exit rate at the lower bound of the quality support:

$$d_t \equiv g_t^Q \lim_{x \to \underline{x}_t} m_t^O(x) / M_t.$$

Hence, the total measure of products evolves according to:

$$\dot{M}_t = (e_t - d_t)M_t$$

where $e_t \equiv R_t / M_t$ denotes the entry rate.

Equilibrium Conditions. Using our previous results, the free-entry condition can be rewritten as:

$$V_t^N \underline{q}^{\theta-1} = (1 - \tau^R)c_R Q_t M_t^{\frac{1}{\theta-1}}$$

the labor market clearing condition can be rewritten as:

$$N_t = \frac{Y_t}{\mu w_t}$$

and the final good market clearing condition can be rewritten as:

$$C_t + c_R Q_t M_t^{\frac{1}{\theta-1}} E_t + \frac{c_D Q_t N_t^{\frac{1}{\theta-1}} \gamma_t^{1+\zeta}}{1+\zeta} \cdot \int_{\underline{x}_t}^{\infty} \exp[(\theta-1)x] m_t^N(x) \mathrm{d}x = Y_t.$$

The only endogenous variable for which a corresponding equation is missing is the growth rate of the average quality index. Using the change of variable defined above, the expression for this index implies:

$$M_t^{-1} \int_{\underline{x}_t}^{\infty} \exp[(\theta-1)x] m_t(x) \mathrm{d}x = 1.$$

Balanced Growth Path. On a BGP, the growth rate of the average quality index and the drift of product quality for new firms are both constant. Moreover, the measure of new and old firms both grow at the same rate as the population. This implies that the entry and exit rates are constant, and the former is equal to $e = n + d$. The share of new and old products are also constant and equal to:

$$\frac{M_t^N}{M_t} = \frac{n+d}{n+\epsilon} \quad \text{and} \quad \frac{M_t^O}{M_t} = \frac{\epsilon - d}{n+\epsilon}.$$

Defining $f_t^N(x) \equiv m_t^N(x)/M_t^N$, the stationary FP equation for this distribution of log relative quality among new firms is:

$$-(\gamma - g^Q)\partial_x f^N(x) - (n+\epsilon) f^N(x) = -(n+\epsilon)\delta(x - \underline{x}).$$

With parameter values such that $\gamma > g^Q$, the solution to this ODE is:

$$f^N(x) = \lambda_N \exp[-\lambda_N(x - \underline{x})] \quad \text{where} \quad \lambda_N \equiv \frac{n+\epsilon}{\gamma - g^Q}.$$

Similarly, defining $f_t^O(x) \equiv m_t^O(x)/M_t^O$, the stationary FP equation for this distribution of log relative quality among old firms is:

$$g^Q \partial_x f^O(x) - n f^O(x) = -\frac{\epsilon(n+d)}{\epsilon - d} \cdot f^N(x).$$

Dividing through by g^Q, and multiplying by the integration factor $\exp(-\lambda_O x)$ where $\lambda_O \equiv n/g_Q$, we can rewrite:

$$\partial_x[\exp(-\lambda_O x) f^O(x)] = -\frac{\epsilon(n+d)}{(\epsilon-d)g^Q} \cdot \exp(-\lambda_O x) f^N(x).$$

Integrating this equation and solving for $f^O(x)$, we obtain:

$$f^O(x) = \left[C - \frac{\epsilon(n+d)}{(\epsilon-d)g^Q} \cdot \int_{\underline{x}}^{x} f^N(x)\exp(-\lambda_O x)\mathrm{d}x\right]\exp(\lambda_O x)$$

where C is an integration constant. For $f^O(x)$ to be integrable, we must have:

$$C = \frac{\epsilon(n+d)}{(\epsilon-d)g^Q} \cdot \int_{\underline{x}}^{\infty} f^N(x)\exp(-\lambda_O x)\mathrm{d}x.$$

Substituting this expression back in the solution for $f^O(x)$, we obtain:

$$f^O(x) = \frac{\epsilon(n+d) f^N(x)}{(\epsilon-d)(\lambda_N g^Q + n)}.$$

For $f^O(x)$ to be a probability distribution, we must verify that:

$$\epsilon(n+d) = (\epsilon-d)(\lambda_N g^Q + n).$$

Using the definition of the exit rate, which delivers $d = \epsilon g^Q/\gamma$, it is straightforward to verify that this condition is satisfied. Therefore, we have that the stationary distribution of quality among old firms is identical to that of new firms. Let us define normalized variables, which are constant on a BGP:

$$\mathcal{V}^O \equiv \frac{V_t^O}{c_t}, \quad \mathcal{V}^N \equiv \frac{V_t^N}{c_t}, \quad \mathcal{M} \equiv \frac{M_t}{N_t}, \quad \mathcal{W} \equiv \frac{w_t}{c_t}, \quad \mathcal{Y} \equiv \frac{y_t}{c_t}.$$

With these definitions, we can express the free-entry condition as:

$$(1-\tau^R)c_R\mathcal{Y}\underline{q}^{1-\theta} = \frac{\epsilon\mathcal{V}^O + (1-\tau^C)\mathcal{Y}/(\theta\mathcal{M}) + (1-\tau^D)c_D\mathcal{Y}\mathcal{M}^{\frac{1}{1-\theta}}\gamma^{1+\zeta}/(1+\zeta)}{\rho+\epsilon+(\theta-1)(g^Q-\gamma)}$$

where we have the following definitions:

$$\mathcal{V}^O = \frac{(1-\tau^C)\mathcal{Y}}{\theta\mathcal{M}[\rho + (\theta-1)g^Q]},$$
$$\gamma = \left[\frac{(\theta-1)(1-\tau^R)c_R\mathcal{M}^{\frac{1}{\theta-1}}}{(1-\tau^D)c_D\underline{q}^{\theta-1}}\right]^{1/\zeta},$$
$$\mathcal{Y} = 1 + c_R\mathcal{Y}(n+d)\mathcal{M} + \frac{c_D\gamma^{1+\zeta}}{1+\zeta}\cdot\frac{n+d}{n+\epsilon}\cdot\mathcal{Y}\mathcal{M}^{\frac{\theta-2}{\theta-1}},$$
$$g^Q = \gamma - \frac{(n+\epsilon)(1-\underline{q}^{\theta-1})}{\theta-1}.$$

To obtain that last expression for the growth rate of the average quality index, we use the definition of the index itself:

$$\frac{\lambda_N\cdot\underline{q}^{\theta-1}}{\lambda_N+1-\theta} = 1 \quad\Leftrightarrow\quad g^Q = \gamma - \frac{(n+\epsilon)(1-\underline{q}^{\theta-1})}{\theta-1}.$$

A.2 The Constrained-Optimal Allocation

Notation. We introduce the following notation to define the inner product between two square-integrable functions $f(x), g(x) : \Omega \to \mathbb{R}$ over their common domain:

$$\langle f(x), g(x)\rangle_{x\in\Omega} \equiv \int_\Omega f(x)g(x)\mathrm{d}x.$$

Second, let us denote the (partial) Gateaux derivative of a functional F with respect to the function $f(x)$ in direction $\tau(x)$ as:

$$\delta F[f(x);\tau(x)] \equiv \left.\frac{\partial F[f(x)+\varepsilon\cdot\tau(x),.]}{\partial\varepsilon}\right|_{\varepsilon=0}$$

where the functional F can take additional arguments through the "dot" notation and the "test" function $\tau(x)$ is assumed to vanish on the boundaries of the relevant integration domain.

The Planner's Problem. Consider the problem of a planner seeking to maximize the following objective:

$$U_0 = \int_0^\infty e^{-(\rho-n)t}\ln(c_t)\mathrm{d}t$$

subject to the constraints:[6]

$$
\begin{aligned}
Y_t &= [\textstyle\sum_{T\in\{N,O\}} \langle (q y_t^T(q))^{\frac{\theta-1}{\theta}}, m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}]^{\frac{\theta}{\theta-1}}, \\
N_t &\geq \textstyle\sum_{T\in\{N,O\}} \langle y_t^T(q), m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}, \\
Y_t &\geq C_t + R_t + \langle d_t(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)}, \\
\dot{m}_t^N(q) &= -\partial_q[\gamma_t(q) q m_t^N(q)] - \epsilon m_t^N(q) + \delta(q-\underline{q}_t) R_t/(c_R Q_t M_t^{\frac{1}{\theta-1}}), \\
\dot{m}_t^O(q) &= \epsilon m_t^N(q)
\end{aligned}
$$

by choosing $\{\{\{y_t^T(q)\}_{T\in\{N,O\}}, d_t(q)\}_{q=\underline{q}_t}^\infty, R_t\}_{t=0}^\infty$. The solution to the planner's problem is "constrained-optimal" in the sense that the planner takes externalities across firms (technology spillovers) as given. More precisely, the planner takes M_t and Q_t as given in the research and development technologies. Reformulating this problem using the current-value Hamiltonian, we obtain:

$$
\begin{aligned}
\mathcal{H}_t &= \ln(c_t) + \nu_t^L[N_t - \textstyle\sum_{T\in\{N,O\}} \langle y_t^T(q), m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}] \\
&+ \nu_t^Y\{[\textstyle\sum_{T\in\{N,O\}} \langle (q y_t^T(q))^{\frac{\theta-1}{\theta}}, m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}]^{\frac{\theta}{\theta-1}} - c_t N_t - R_t - \langle d_t(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)}\} \\
&- \langle \nu_t^N(q), \partial_q[\gamma_t(q) q m_t^N(q)]\rangle_{q\in[\underline{q}_t,\infty)} + \epsilon\langle \nu_t^O(q) - \nu_t^N(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)} \\
&+ \nu_t^N(\underline{q}_t) R_t/(c_R Q_t M_t^{\frac{1}{\theta-1}})
\end{aligned}
$$

where $\{\nu_t^L, \nu_t^Y, \{\nu_t^N(q), \nu_t^O(q)\}_{q=\underline{q}_t}^\infty\}_{t=0}^\infty$ are the costate functions. Using integration by parts, we can rewrite:

$$
\begin{aligned}
\mathcal{H}_t &= \ln(c_t) + \nu_t^L[N_t - \textstyle\sum_{T\in\{N,O\}} \langle y_t^T(q), m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}] \\
&+ \nu_t^Y\{[\textstyle\sum_{T\in\{N,O\}} \langle (q y_t^T(q))^{\frac{\theta-1}{\theta}}, m_t^T(q)\rangle_{q\in[\underline{q}_t,\infty)}]^{\frac{\theta}{\theta-1}} - c_t N_t - R_t - \langle d_t(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)}\} \\
&+ \langle \gamma_t(q) q \partial_q \nu_t^N(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)} + \epsilon\langle \nu_t^O(q) - \nu_t^N(q), m_t^N(q)\rangle_{q\in[\underline{q}_t,\infty)} \\
&+ \nu_t^N(\underline{q}_t) R_t/(c_R Q_t M_t^{\frac{1}{\theta-1}}).
\end{aligned}
$$

The first-order condition with respect to $y_t^T(q)$ implies:

$$
\nu_t^Y \delta Y_t[y_t^T(q); \varrho(q)] = \nu_t^L \langle m_t^T(q), \varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}
$$

[6]Here, we substituted the product resource constraints in the labor resource constraint.

where $\delta Y_t[y_t^T(q);\varrho(q)]$ is the Gateaux derivative of Y_t with respect to $y_t^T(q)$ in direction $\varrho(q)$, which is an arbitrary function that vanishes on the boundaries of $[\underline{q}_t,\infty)$:

$$\delta Y_t[y_t^T(q);\varrho(q)] = Y_t^{1/\theta}\langle q^{\frac{\theta-1}{\theta}} y_t^T(q)^{-1/\theta} m_t^T(q),\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}.$$

Since that first-order condition must hold for any function $\varrho(q)$, we obtain the relative demand functions:

$$y_t^T(q)/Y_t = q^{\theta-1}(\nu_t^Y/\nu_t^L)^\theta,\ \ \forall q\in[\underline{q}_t,\infty).$$

Integrating this expression, we find:

$$\nu_t^L = \nu_t^Y M_t^{\frac{1}{\theta-1}} Q_t$$

such that we can rewrite:

$$y_t^T(q) = \frac{\nu_t^Y Y_t (q/Q_t)^{\theta-1}}{\nu_t^L M_t},\ \ \forall q\in[\underline{q}_t,\infty).$$

The first-order condition with respect to $d_t(q)$ implies:

$$\langle \gamma_t(q) q\partial_q \nu_t^N(q) m_t^N(q)/d_t(q),\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}/(1+\zeta) = \nu_t^Y\langle m_t^N(q),\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}.$$

Since that first-order condition must hold for any function $\varrho(q)$, we obtain:

$$\gamma_t(q) = \left[\frac{q\partial_q \nu_t^N(q)}{\nu_t^Y c_D Q_t N_t^{\frac{1}{\theta-1}}(q/Q_t)^{\theta-1}}\right]^{1/\zeta},\ \ \forall q\in[\underline{q}_t,\infty).$$

The first-order condition with respect to R_t implies:

$$\nu_t^Y c_R Q_t M_t^{\frac{1}{\theta-1}} = \nu_t^N(\underline{q}_t).$$

The first-order condition with respect to $m_t^N(q)$ implies:

$$\begin{aligned}&\langle(\rho-n)\nu_t^N(q) - \dot{\nu}_t^N(q),\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)} = \langle (q/Q_t)^{\theta-1},\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}\nu_t^Y Y_t/[(\theta-1)M_t]\\ &+\langle\gamma_t(q)q\partial_q\nu_t^N(q) - \nu_t^Y d_t(q) + \epsilon[\nu_t^O(q)-\nu_t^N(q)],\varrho(q)\rangle_{q\in[\underline{q}_t,\infty)}.\end{aligned}$$

Since this condition must hold for any function $\varrho(q)$, we have:

$$(\rho - n)\nu_t^N(q) - \dot{\nu}_t^N(q) = (q/Q_t)^{\theta-1}\nu_t^Y Y_t/[(\theta-1)M_t] - \nu_t^Y d_t(q)$$
$$+ \gamma_t(q)q\partial_q \nu_t^N(q) + \epsilon[\nu_t^O(q) - \nu_t^N(q)], \;\; \forall q \in [\underline{q}_t, \infty).$$

Similarly, the first-order condition with respect to $m_t^O(q)$ implies:

$$(\rho - n)\nu_t^O(q) - \dot{\nu}_t^O(q) = (q/Q_t)^{\theta-1}\nu_t^Y Y_t/[(\theta-1)M_t], \;\; \forall q \in [\underline{q}_t, \infty).$$

Defining the following functions:

$$V_t^{N*}(q) \equiv \nu_t^N(q)/(\mu\nu_t^Y), \;\; V_t^{O*}(q) \equiv \nu_t^O(q)/(\mu\nu_t^Y), \;\; w_t^* \equiv \nu_t^L/(\mu\nu_t^Y), \;\; r_t^* \equiv \dot{c}_t/c_t + \rho,$$

and substituting them in the new firm's social HJB equation, we obtain:

$$r_t^* V_t^{N*}(q) - \dot{V}_t^{N*}(q) = (q/Q_t)^{\theta-1} Y_t/(\theta M_t) - d_t(q)/\mu$$
$$+ \gamma_t(q)q\partial_q V_t^{N*}(q) + \epsilon[V_t^{O*}(q) - V_t^{N*}(q)]$$

where $\gamma_t(q)$ is given by:

$$\gamma_t(q) = \left[\frac{\mu q \partial_q V_t^{N*}(q)}{c_D Q_t N_t^{\frac{1}{\theta-1}}(q/Q_t)^{\theta-1}}\right]^{1/\zeta}.$$

Doing so for the old firm's social HJB equation, we obtain:

$$r_t^* V_t^{O*}(q) - \dot{V}_t^{O*}(q) = (q/Q_t)^{\theta-1} Y_t/(\theta M_t).$$

Finally, substituting these functions in the social free-entry condition and the labor resource constraint, we obtain:

$$\frac{c_R Q_t M_t^{\frac{1}{\theta-1}}}{\mu} = V_t^{N*}(\underline{q}_t) \quad \text{and} \quad N_t = \frac{Y_t}{\mu w_t^*}.$$

This demonstrates that the constrained-optimal allocation can be implemented with research and development subsidies of $\tau^R = \tau^E = 1/\theta$.