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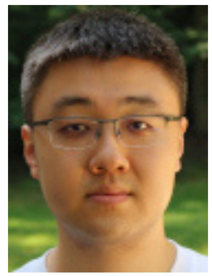
IHES
Institut des Hautes Études Scientifiques

Brownian loop measure, det Δ, and length spectra of Riemann surfaces

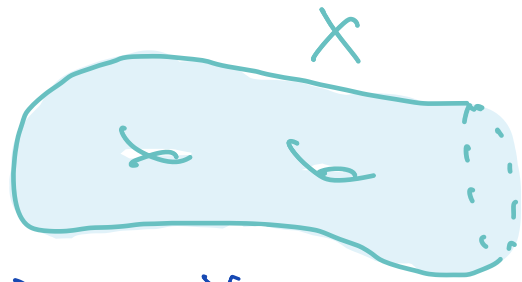
Yilin Wang (IHES)

Joint with Yuhao Xue
(IHES)



Collège de France 01.2025

Riemannian surface (X, g)



Assumption:

- X without boundary
  (∂X is not included in X)
- g a smooth Riemannian metric on X

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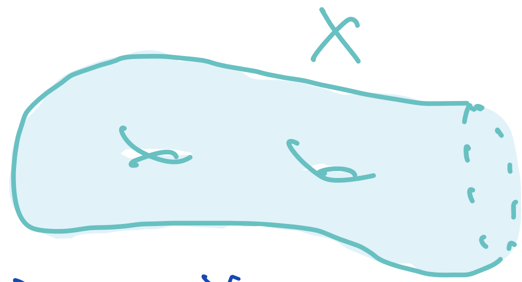

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→ X may have punctures

→ X does not need to be geometrically finite
(∞ genus, ∞ punctures allowed)

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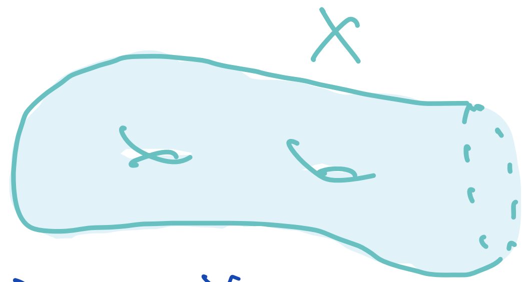

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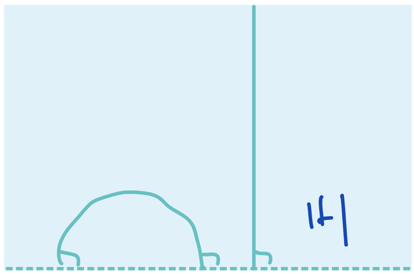
→ X admits a metric of form $e^{2\sigma} g$ such that $(X, e^{2\sigma} g)$ has constant curvature $\in \{1, 0, -1\}$ and is complete. (Poincaré, Koebe)

- 1: $X = S^2$
- 0: $X = \mathbb{C}$, $\mathbb{C} \setminus \{0\}$, torus $\mathbb{C}/\Lambda$
- -1: $X =$ anything else ← hyperbolic, unique

Riemann Surface: surface / complex structure = conformal class of metric

Complete metric examples

$z = x + iy$



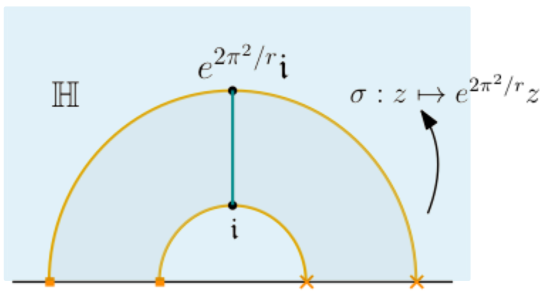
geodesics

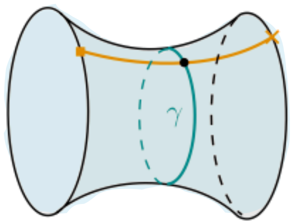
$g = dx^2 + dy^2$ not complete

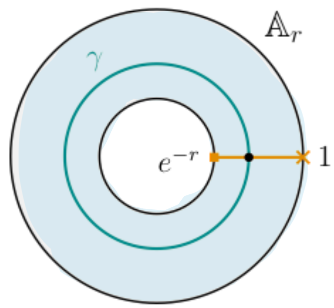
$g_{\mathbb{H}} = \dfrac{dx^2 + dy^2}{y^2}$ complete

curvature $= -1$

hyperbolic plane







hyperbolic annulus

Complete hyperbolic surface

(X, g_0) has curvature $\equiv -1$

$\Gamma \triangleleft PSL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$

discrete subgroup without torsion

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$

$z \mapsto \dfrac{az+b}{cz+d}$ that does not have fixed point in $\mathbb{H}$

Complete hyperbolic surface

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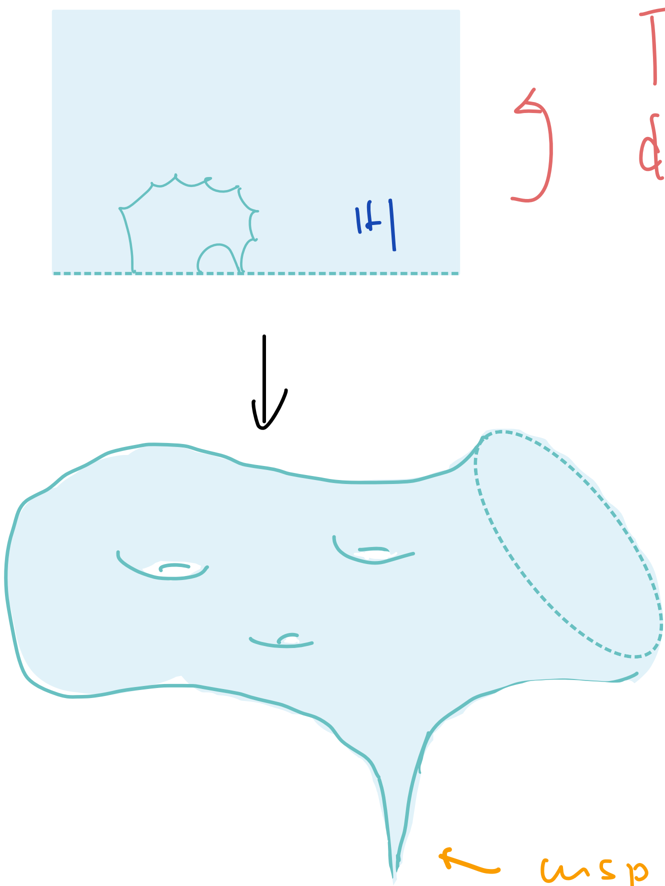


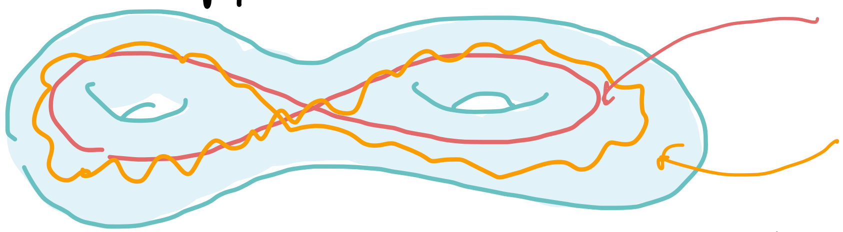
$\Gamma \backslash \mathbb{H} = X$, $\Gamma \cong \pi_1(X)$

funnel

both ∞-ly far

cusp

Free homotopy classes of closed curve



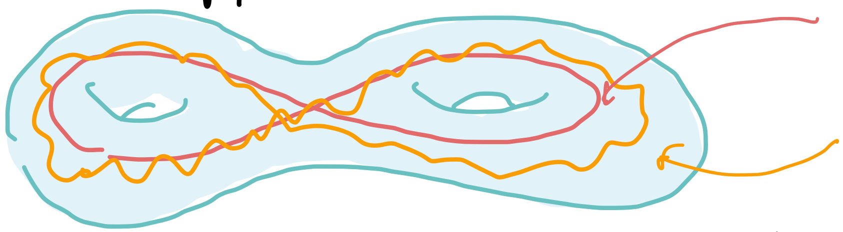
free homotopic

Fact: • If X is hyperbolic, $\forall$ (free) homotopy class of curves not homotopic to a point

(trivial or to a cusp)

$\exists!$ hyperbolic geodesic γ

Free homotopy classes of closed curve



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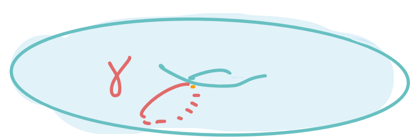
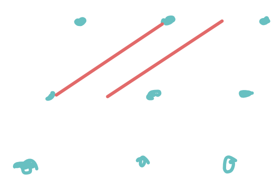
• If T is a flat torus $\mathbb{C}/\Lambda$





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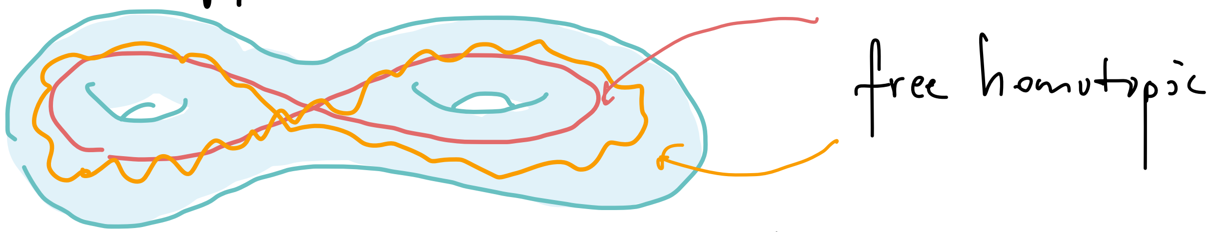

Fact: • If X is hyperbolic, $\forall$ (free) homotopy class of curves not homotopic to a point

$\leftrightarrow$ (trivial or to a cusp)

$\exists!$ hyperbolic geodesic γ $\quad \ell_X(\gamma) \leftarrow$ length

$\downarrow$

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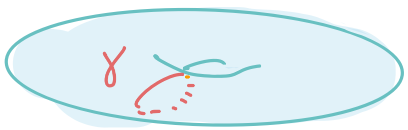
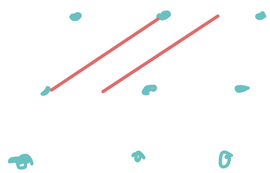
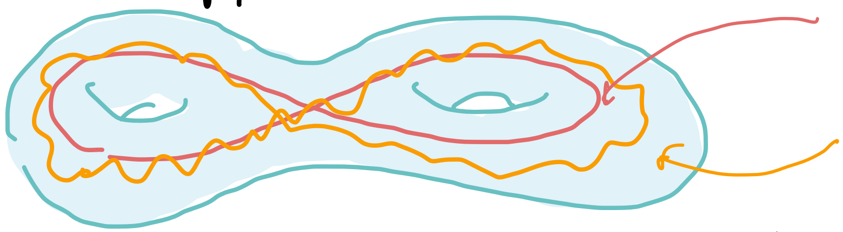




$\ell_T(\gamma)$

• Otherwise, all γ is homotopic to a point.

Free homotopy classes of closed curve



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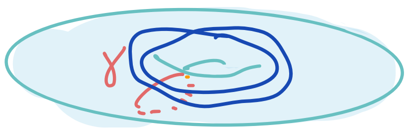
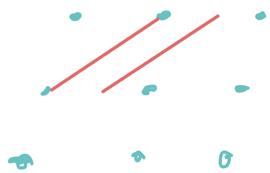




$\downarrow$

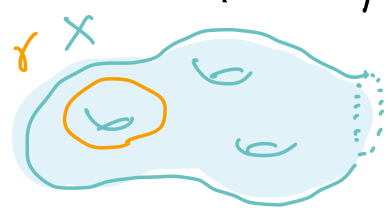
$\ell_T(\gamma)$

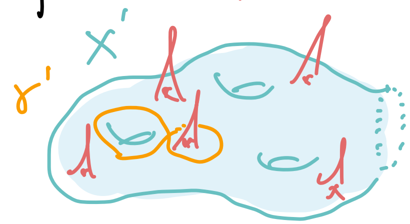
• Otherwise, all γ is homotopic to a point.

• $m_X(\gamma)$ = multiplicity of γ $\quad m_X(\gamma)=1 \Leftrightarrow \gamma$ primitive

Main theorem 1 (W. - Xue)

Let X be a complete hyperbolic surface, $P = \{p_1, \cdots\} \subset X$





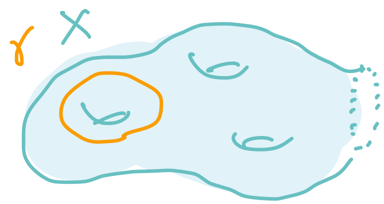
be a closed, countable subset of X. $X' := X \setminus P$ with γ a hyperbolic geodesic in X. We have:

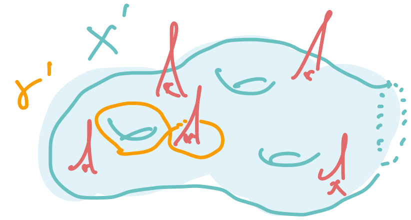
$$\frac{1}{m_X(\gamma)} \frac{1}{e^{\ell_X(\gamma)} - 1} = \sum_{\gamma' \sqsupseteq_X \gamma} \frac{1}{m_{X'}(\gamma')} \frac{1}{e^{\ell_{X'}(\gamma')} - 1}.$$

If T is a flat torus. $X' = T \setminus P$ ...

$$\frac{\mathrm{Area}(T)}{\pi\, \ell_T(\gamma)^2} = \sum_{\gamma' \sqsupseteq_X \gamma} \frac{1}{m_{X'}(\gamma')} \frac{1}{e^{\ell_{X'}(\gamma')} - 1}.$$

Comments





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...

Remark: If γ is primitive $(m_X(\gamma)=1)$, then $m_{X'}(\gamma')=1$.
since $m_{X'}(\gamma') \mid m_X(\gamma)$.

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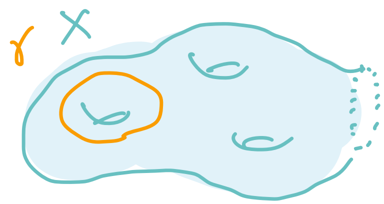

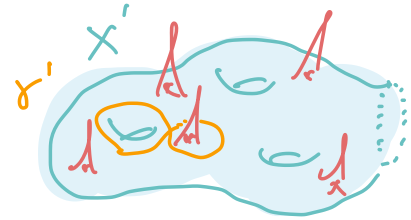

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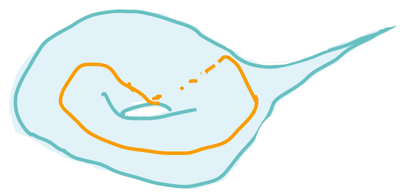
Thm (McShane's identity Invent. Math. 98)

$\forall$ $X = T \setminus \{p_1\}$ once punctured torus.

$$\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)}+1} = 1$$

γ oriented simple closed geodesic



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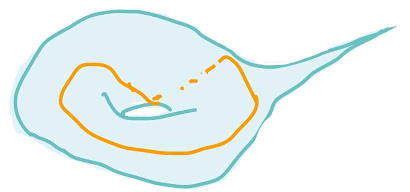
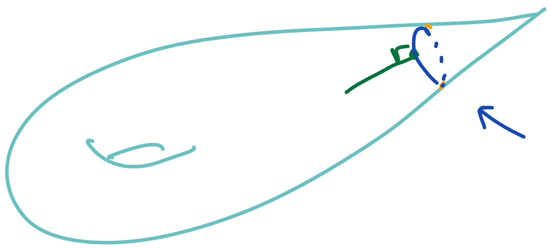
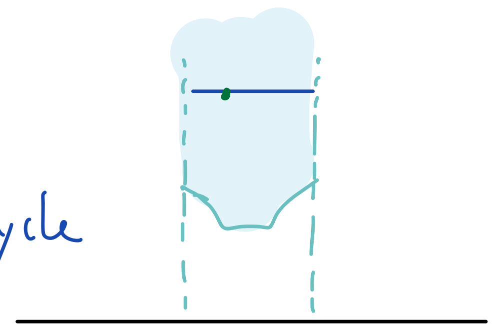


Proof sketch:



a horocycle



pick a point uniformly on horocycle

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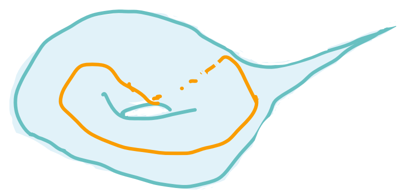

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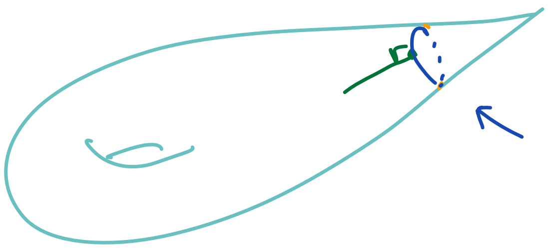

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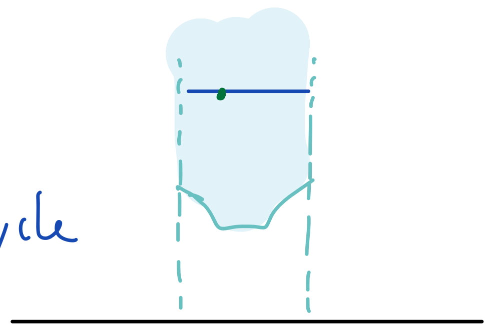
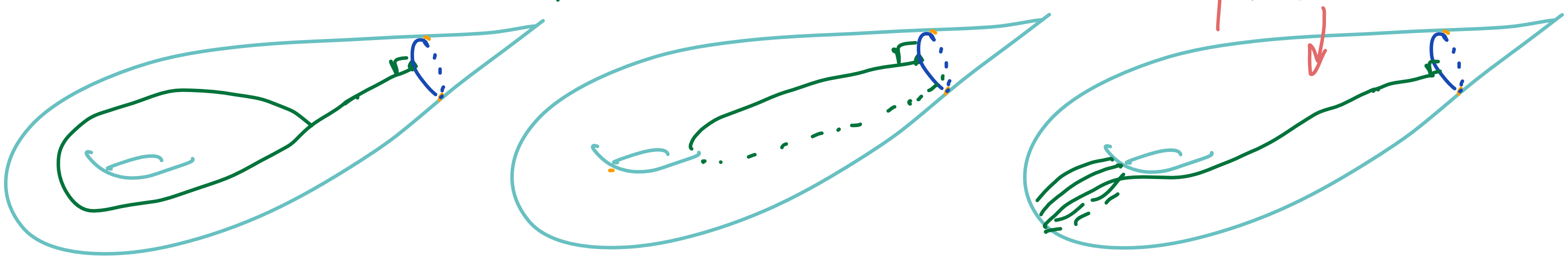


pick a point uniformly on horocycle

prob = 0



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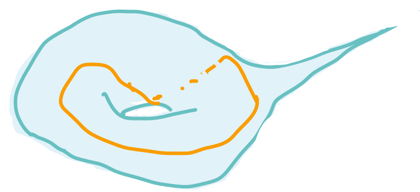

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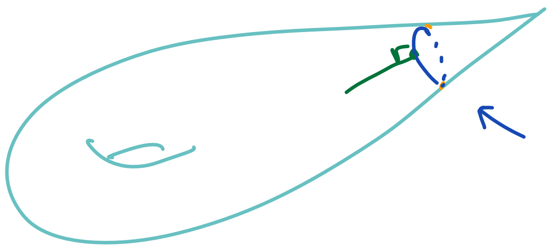

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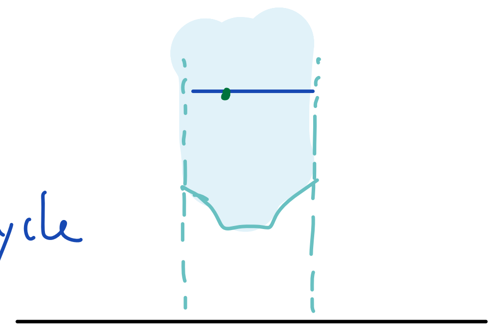

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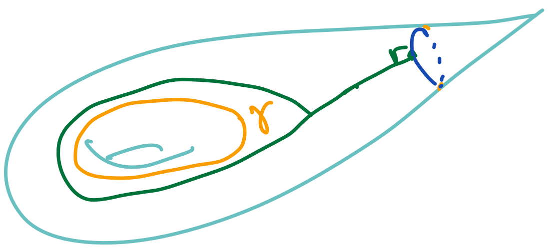
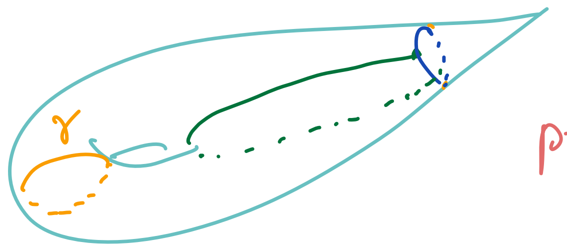




$$\mathrm{prob}(\gamma) = \frac{1}{e^{\ell_X(\gamma)}+1}$$

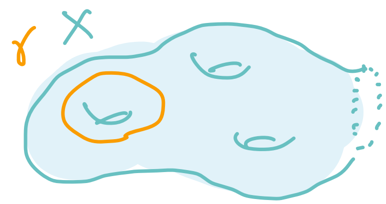
Other identities on length spectrum

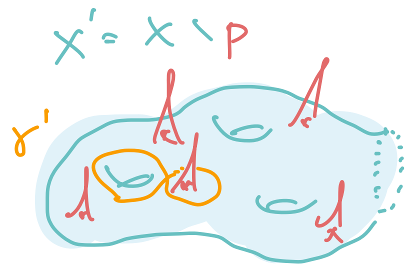
[Mirzakhani] Invent. 07 Generalized McShane's identity $\Rightarrow$ Computation of the Weil-Petersson volume of moduli spaces.

[Basmajian] AJM 93 Relation length of geodesic boundary $\Leftrightarrow$ lengths of orthogeodesics

[Bridgeman] G&T 11 Identities on orthogeodesics by decomposing T^1M

Also [Bridgeman-Kahn] [Calegari] [Parlier] ...





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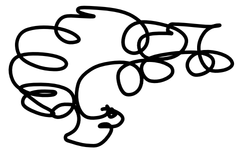
Proof idea:

→ Use Brownian motion on (X, g)

→ Conformal invariance of 2D BM

→ Probability of 2D BM hitting $\{x\}$ is 0

Brownian loop measure



- ∞, σ-finite measure $\mu_{X,g}$ on Brownian type loops on (X,g)
- Introduced by Lawler - Werner to study SLE in $\mathbb{C}$
- Brownian loop measure $\rightarrow$ SET of SLE measures ...

  Based on conformal restriction property of SLE

  [Lawler - Schramm - Werner]

- In $\mathbb{D}$, ppp of intensity $\frac{c(\kappa)}{2}\mu_{\mathbb{D},dz^2}$, outerboundary of cluster

  [Sheffield - Werner] $\rightarrow$ CLE_κ loop ensemble $\kappa \in (\frac{8}{3}, 4]$

- "Total mass = $\log\det \Delta_{X,g}$"

  $\hookrightarrow$ Partition function of GFF

  $\hookrightarrow$ conformal anomaly of CFT

  [Le Jan] [Dubédat] [Ang - Park - Pfeffer - Sheffield]

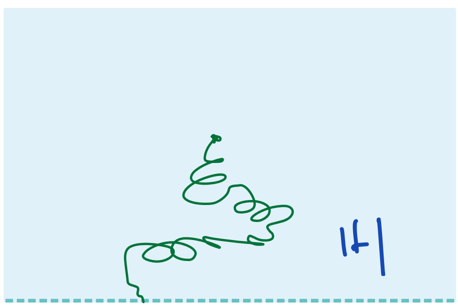
Brownian loop measure

Brownian motion on (X,g) is the diffusion generated by Δ_g (Laplace–Beltrami operator) killed at ∂X.

- In $\mathbb{R}^2$, $g = e^{2\sigma}(dx^2+dy^2)$ $\Rightarrow$ $\Delta_g = e^{-2\sigma}(\partial_{xx}+\partial_{yy})$

  $\Rightarrow$ B is a time changed standard Brownian motion

  $\Rightarrow$ Trajectory conformally invariant if killed at ∂X



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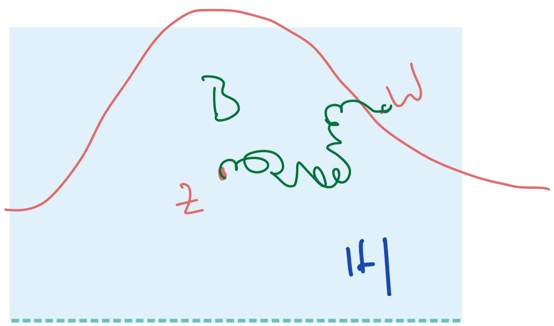

$p_t(z, w)$: heat kernel

$p_t(z, w)\, d\mathrm{vol}(w)$ = distribution of B_t starting at z

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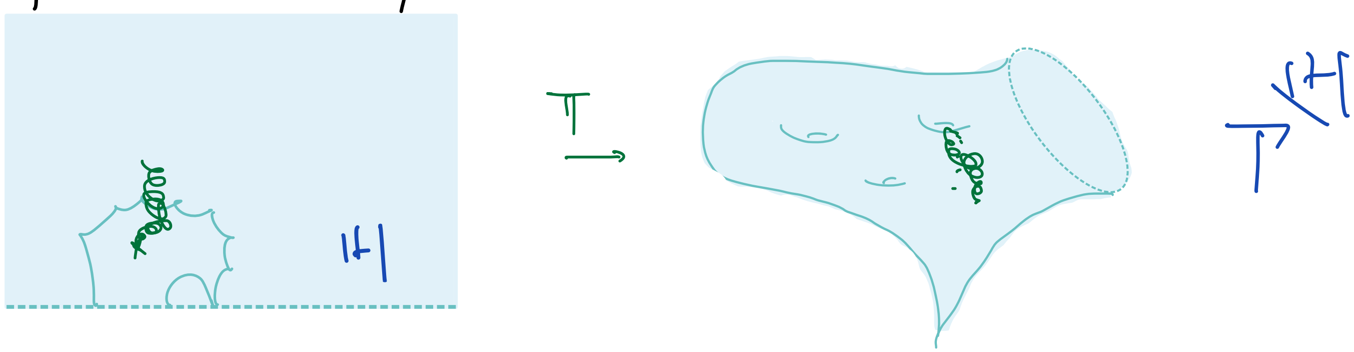
- Project to X hyperbolic



Definition : Brownian loop measure on (X, g)

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- $\mathbb{W}_{x\to y}^t$ such that

$$\mathbb{W}_x^t = \int_{y \in X} \mathbb{W}_{x\to y}^t \, d\mathrm{vol}_g(y)$$

Remark : $|\mathbb{W}_{x\to y}^t| = P_t(x,y)$ heat kernel with Dirichlet ∂- condition

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- Brownian loop measure

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}_{x \to x}^t \, dvol_g(x) \, dt$$

Properties of BLM

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}^t_{x\to x} \, d\mathrm{vol}_g(x) \, dt$$

- Infinite measure on loops in X
- Restriction: If $X' \subset X$, $\mu_{X',g} = \mu_{X,g} \mathbb{1}_{\{\cdot \subset X\}}$.
- Conformal invariance: $\forall \sigma \in C^\infty(X, \mathbb{R})$

$$\mu_{X,g} = \mu_{X, e^{2\sigma} g}$$

(forgetting the parametrization but keeping the orientation)

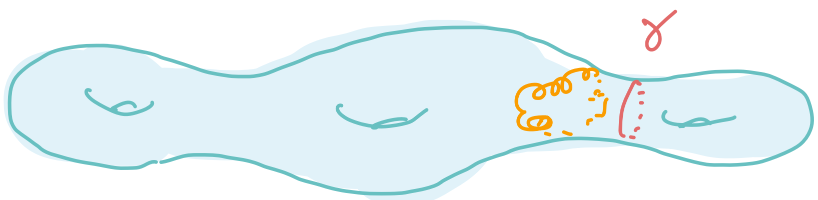
Properties of BLM

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}^t_{x\to x}\, d\mathrm{vol}_g(x)\, dt$$

- Infinite measure on loops in X
- Restriction: If $X' \subset X$, $\mu_{X',g} = \mu_{X,g} \mathbb{1}_{\{\cdot \subset X'\}}$.
- Conformal invariance: $\forall \sigma \in C^\infty(X, \mathbb{R})$

$$\mu_{X,g} = \mu_{X, e^{2\sigma} g}$$

(forgetting the parametrization but keeping the orientation)

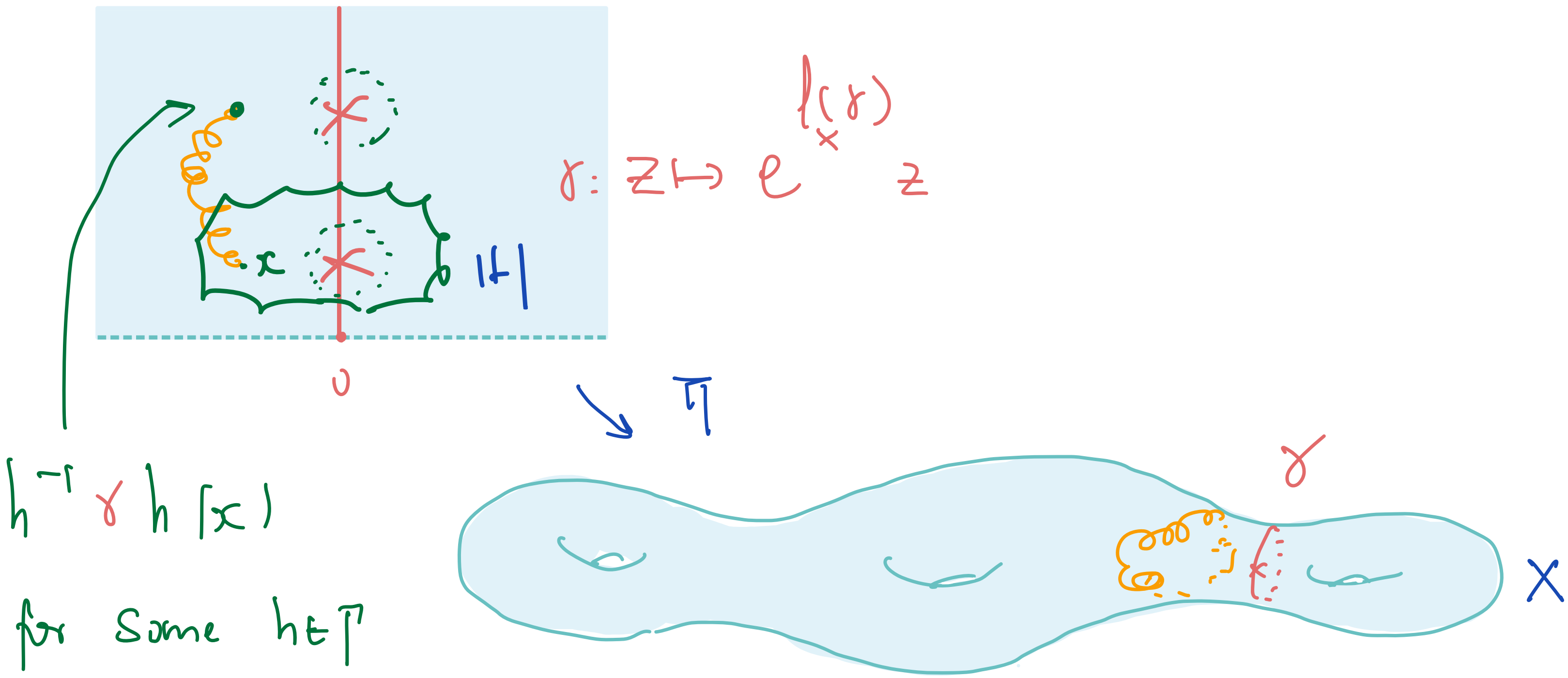


Expect loops in a non-trivial free homotopic class

$\mu_{X,g}([\gamma]) < \infty$ [Le Jan]

Computing $\mu_{x,g}([\gamma])$

- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.



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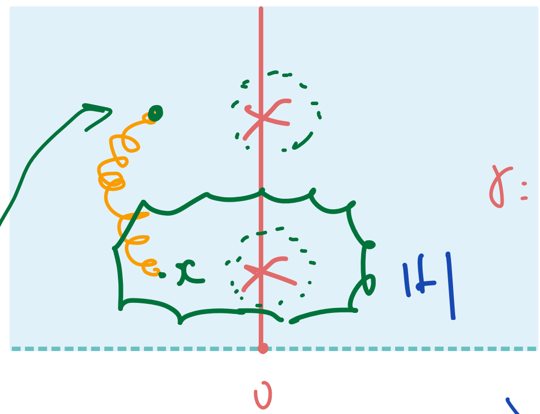
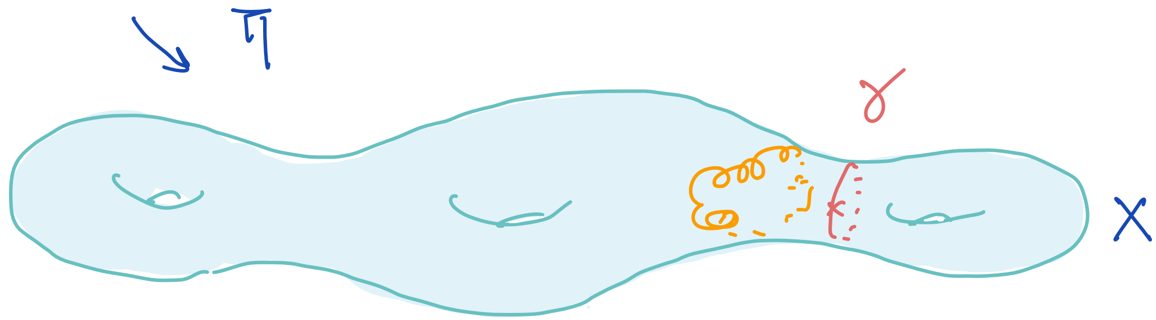


$\gamma : z \mapsto e^{\ell_x(\gamma)} z$



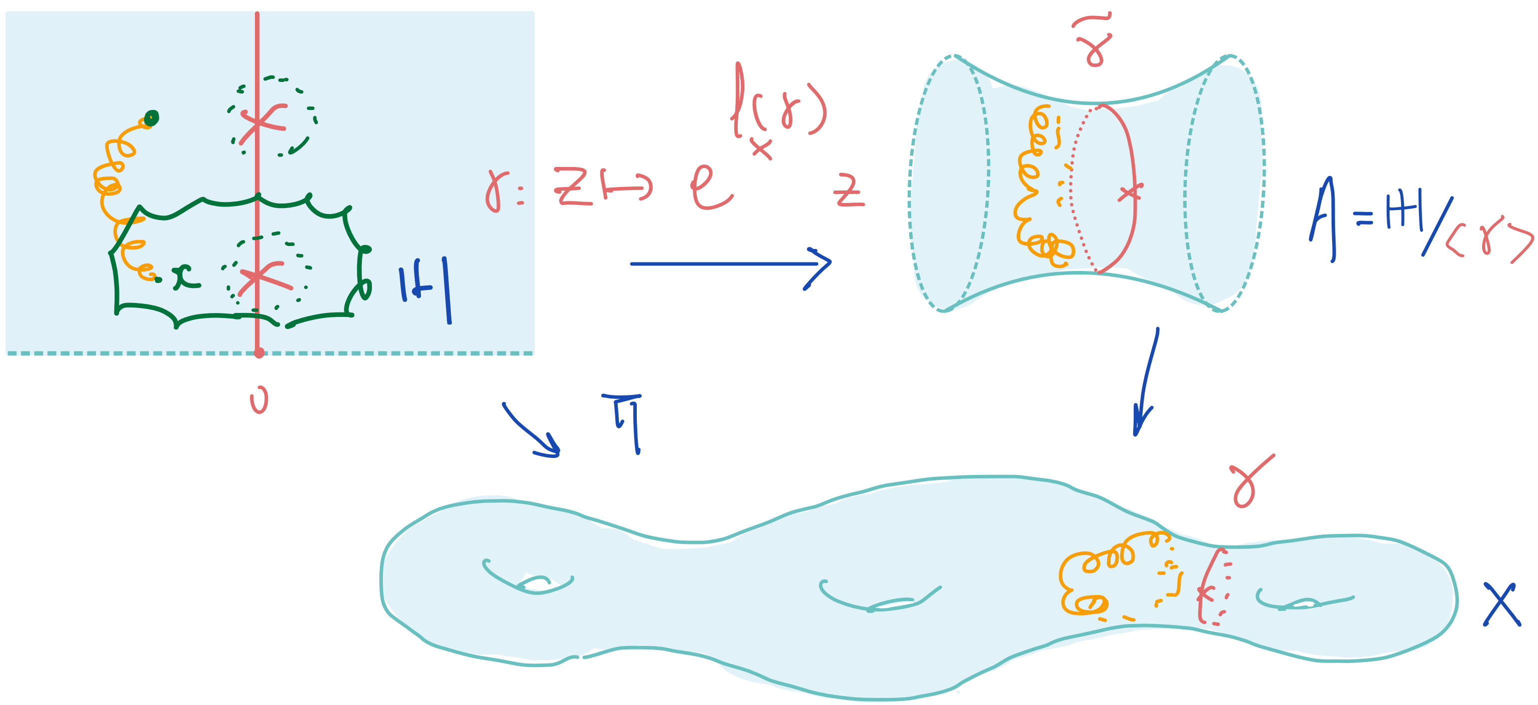
$\underbrace{h^{-1}\gamma h}_{\gamma'}(x)$

for some $h \in \Gamma$

$$\mu_x([\gamma]) = \int_0^\infty \frac{1}{t} \int_F \sum_{\gamma' \in \text{conjugacy class of } \gamma \text{ in } \Gamma} \underbrace{P_t^{\mathbb{H}}(x, \gamma' x)}_{" P_t^{\mathbb{H}}(h(x),\, \gamma h(x)) "} \, d\mathrm{vol}_x \, dt$$

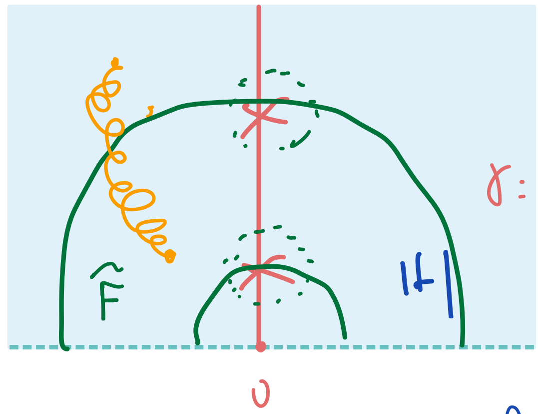
Computing $\mu_{X,g}([\gamma])$

- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.

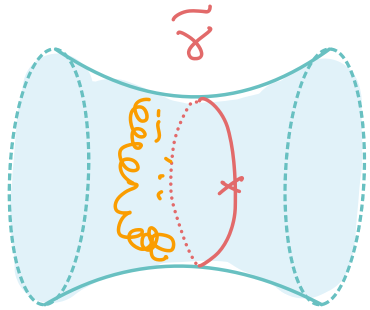


Alternatively: Show $\mu_X([\gamma]) = \mu_A([\tilde{\gamma}])$.

Step 2: Compute $\mu_A([\tilde{\gamma}])$



$\gamma : z \mapsto e^{\ell} z$



$A = \mathbb{H}/\langle \gamma \rangle$

$$\mu_A([\tilde{\gamma}]) = \int_0^\infty \frac{1}{t} \int_{x \in \tilde{F}} p_t^{\mathbb{H}}(x, e^{\ell} x)\, d\mathrm{vol}_{\mathbb{H}}(x)\, dt$$

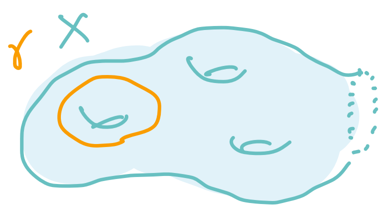
$$= \frac{1}{m_A(\tilde{\gamma})} \frac{1}{e^{\ell_A(\tilde{\gamma})} - 1}$$

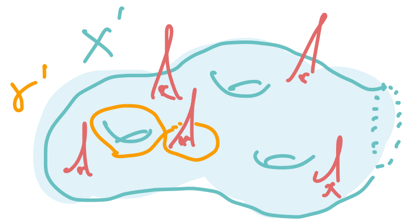
$$\Rightarrow \mu_X([\gamma]) = \frac{1}{m_X(\gamma)} \frac{1}{e^{\ell_X(\gamma)} - 1}$$

$\boxtimes$

Proof of main theorem 1

$X' = X \setminus p$





$$\underbrace{\frac{\mathrm{Area}(T)}{\pi\, \ell_T(\gamma)^2} \text{ or } \frac{1}{m_X(\gamma)}\,\frac{1}{e^{\ell_X(\gamma)}-1}}_{\mu_X([\gamma])} = \underbrace{\sum_{\gamma' \simeq_X \gamma} \frac{1}{m_{X'}(\gamma')}\,\frac{1}{e^{\ell_{X'}(\gamma')}-1}}_{\dots} .$$

$$\mu_X([\gamma]) = \sum_{\gamma' \simeq_X \gamma} \mu_{X'}([\gamma'])$$

p is polar for BM

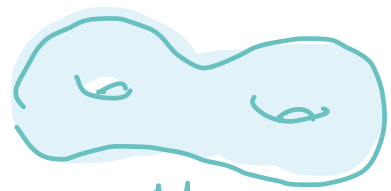
μ_X is conformally invariant

Total mass of Brownian loops

$\to$ Determinant of Δ

Determinant of Laplacian Δ

- M complete hyperbolic closed surface, genus $< \infty$.
- The spectrum is
$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$$



M

- Define the zeta-function
$$\zeta_\Delta(s) := \sum_{i \geq 1} \lambda_i^{-s},$$
it can be analytically continued to a neighborhood of 0.
- Define (following Ray & Singer 1976)
$$\log \det{}'_\zeta(\Delta_g(S^2)) := -\zeta'_\Delta(0)$$
$$\text{“} = \sum_{i \geq 1} \log(\lambda_i)\lambda_i^{-s}|_{s=0} = \log(\prod_{i \geq 1} \lambda_i).\text{”}$$
- The Zeta-regularization of determinants has been used by physicists to perform Feynman path integrals, and is also important in Polyakov's quantum bosonic string theory as being the partition function of Gaussian free field.

Heuristics

$$\zeta_\Delta(s) = \sum_{i\geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\mathrm{Tr}(e^{t\Delta}) - 1\right) dt$$

$$\text{“} \underset{s\to 0}{\sim} s \int_0^\infty t^{-1}\left(\int_M P_t(x,x)\, d\mathrm{vol}(x) - 1\right) dt \text{”}$$

$$-\log \det\nolimits_\zeta(\Delta) = \text{“} \zeta_\Delta'(0) = \underbrace{\int_0^\infty t^{-1} \int_M P_t(x,x)\, d\mathrm{vol}(x)\, dt}_{=\infty} = |\mu_M| \text{”}$$

→ because of small & big loops.

[Ang. Park. Pfeffer. Sheffield] also [LeJan] [Dubédat]

Theorem 4.10 (See [1, Thm. 1.3]). *For a closed Riemannian surface (X, g), the total mass of Brownian loops with quadratic variation greater than $4t$ and less than $4T$ is given by*

$$\frac{\mathrm{Vol}_g(X)}{4\pi t} - \log \det\nolimits_\zeta \Delta_g - \frac{\chi(X)}{6}(\log t + \gamma) + \log T + \gamma + O(t) + O(e^{-\alpha T})$$

as $t \to 0$ and $T \to \infty$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, $\chi(X) = 2 - 2\,\mathrm{genus}(X)$ is the Euler characteristic of X, and $\alpha > 0$.

Normalization by homotopy classes

$\mathcal{G}(X) = \{$ geodesics of $X\}$. $\mathcal{P}(X)$: $\{$primitive geod. in $X\}$

Prime geodesic theorem (Huber, Randol)

$$\underbrace{N_X(L)}_{\#\{\gamma \in \mathcal{P}(X) \mid \ell_X(\gamma) \leq L\}} = \mathrm{Li}(\underbrace{e^L}_{e^L/L}) + \sum_{0<\lambda_j\leq\frac{1}{4}} \mathrm{Li}(e^{s_j L}) + O_X(e^{\frac{3}{4}L}/L)$$

where $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$

$s_j = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}$

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where $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$

$s_j = \frac12 + \sqrt{\frac14 - \lambda_j}$

Main Theorem 2 (W.-Xue)

$$-\log \det_\zeta \Delta = -\mathrm{Area}(X)\, E + C + \sum_{\gamma \in \mathcal{G}(X)\setminus\mathcal{P}(X)} \mu_X([\gamma])$$

$$+ \int_{L=0}^{\infty} \frac{1}{e^L - 1}\, d\left(N_X(L) - \mathrm{Li}(e^L)\right).$$

C, E are universal constants

Key: $\zeta_\Delta(s) = \sum_{i \geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\mathrm{Tr}(e^{t\Delta}) - 1\right) dt$

Selberg trace formula

$$\sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j} = \mathrm{Area}(X) \frac{\mathrm{e}^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{r \mathrm{e}^{-r^2/(4t)}}{\sinh(r/2)} \mathrm{d}r + \underbrace{\sum_{\gamma \in \mathcal{P}(X)} \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-t/4}}{(4\pi t)^{1/2}} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} \mathrm{e}^{-\frac{(m\ell(\gamma))^2}{4t}}}_{S_X(t) = \sum_{\gamma \in \mathcal{G}(X)} \int_X \mathbb{W}^t_{x \to x}([\gamma])\, d\mathrm{vol}_x}.$$

Key: $\zeta_0(s) = \sum_{i \geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{Tr}(e^{t\Delta}) - 1)\, dt$

Selberg trace formula

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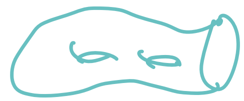
$$-\log \det{}_\zeta \Delta = -\mathrm{Area}(X)E - \gamma + \int_0^1 \frac{S_X(t)}{t} \mathrm{d}t + \int_1^\infty \frac{S_X(t) - 1}{t} \mathrm{d}t$$

Separate $\gamma \in \mathcal{P}(X)$ and $\gamma \in \mathcal{G}(X) \setminus \mathcal{P}(X)$

+ calculation ...

Surface "with boundary"

X : complete hyperbolic surface with $|\chi(X)|<\infty$ and cannot be obtained from removing finitely many points from a closed surface



Prime geodesic theorem (Huber)

$\exists\ \delta \in (0,1)$

$$N_X(L) \sim e^{\delta L}/\delta L \quad \text{as } L \to \infty$$

$\delta = \mathrm{Hdim}(\Lambda(\Gamma))$

$$\Rightarrow \quad \sum_{[\gamma]} \mu_X([\gamma]) < \infty .$$

No normalization needed.

$= ?$

Conclusion

Mass of Brownian loop in a homotopy class $[\gamma]$ on a surface (X, g) is expressed as a function of $\ell_X(\gamma)$ length of a geodesic repr. for the complete metric g_0 of constant curvature and conformal to g.

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$\rightsquigarrow$ Tool to study the length spectrum of Riemann surfaces.

$\Rightarrow$ Identity on length spectra of X & $X \setminus p$

$\Rightarrow$ Another renormalization of total mass of BL (expressed using length) $\leftrightsquigarrow$ $\log\det\Delta$

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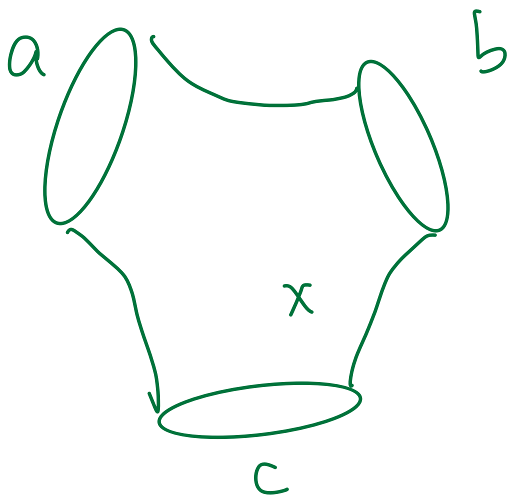
Merci !!!

Remark: BLM is defined for any smooth metric but $\mu_x([\gamma])$ is a function of $\ell_x(\gamma)$ of complete constant curvature metric.

Example:



geodesic boundary (a, b, c)

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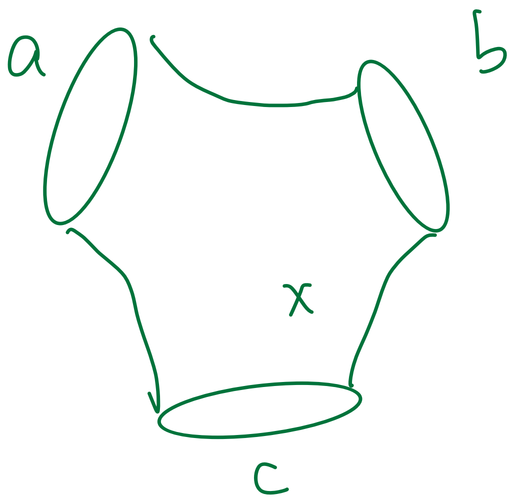
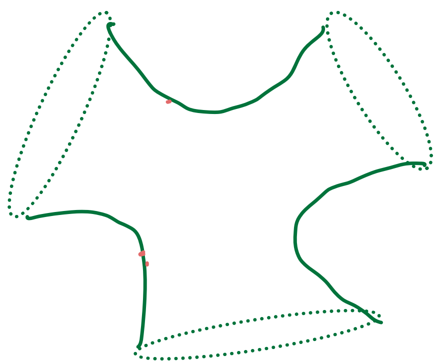


geodesic boundary (a, b, c)

$\updownarrow$

complete surface



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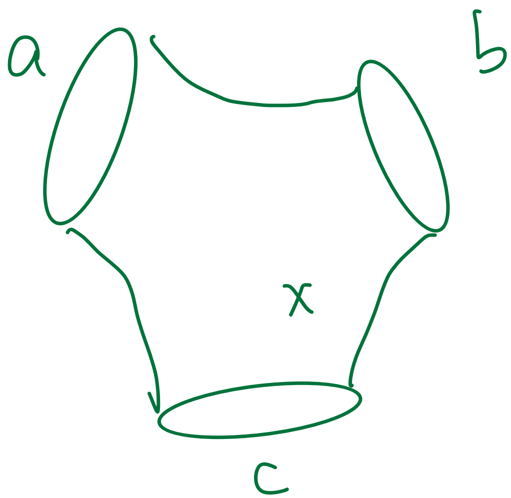

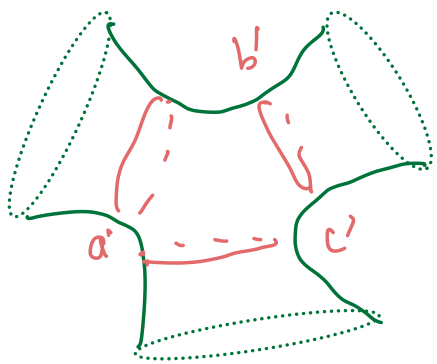

geodesic boundary	(a,b,c)
$\updownarrow$	$\updownarrow$??
complete surface	(a',b',c')

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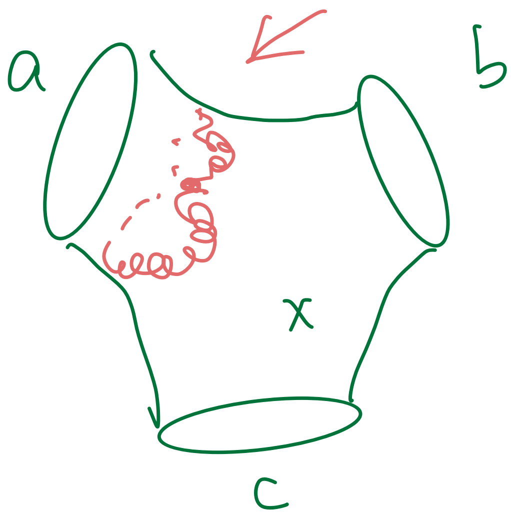
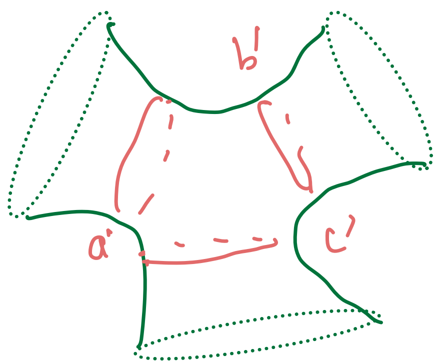
Example:

$$\frac{1}{e^{a'}-1} = \mu_x([a])$$





geodesic boundary (a, b, c)

$$\updownarrow \qquad \overset{??}{\updownarrow}$$

complete surface (a', b', c')