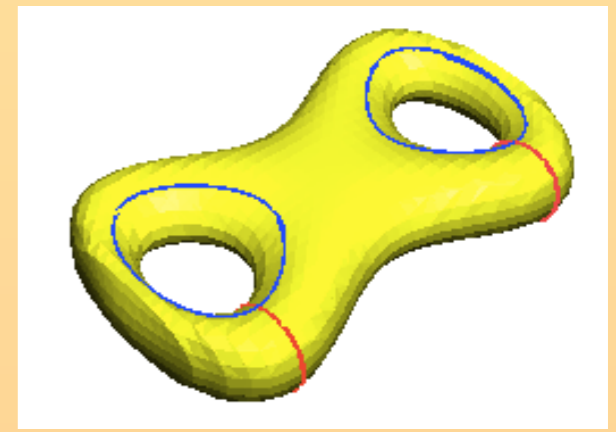
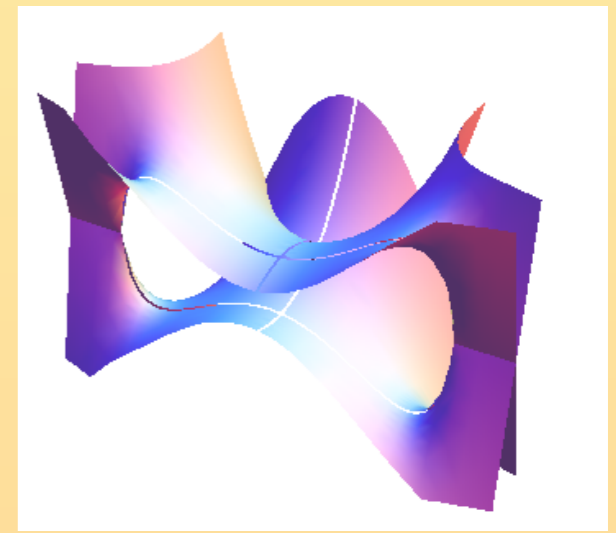
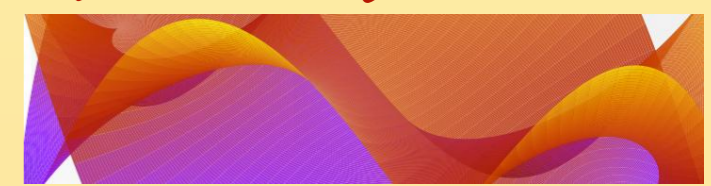
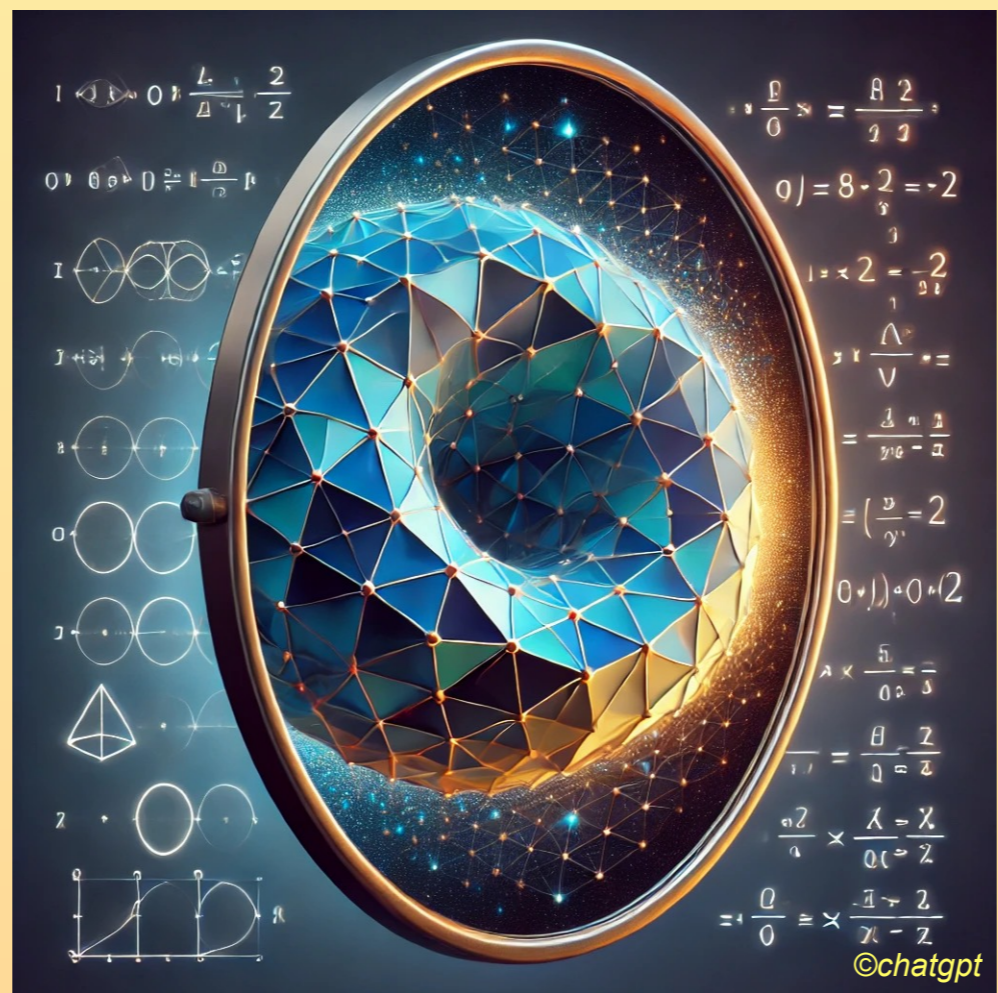
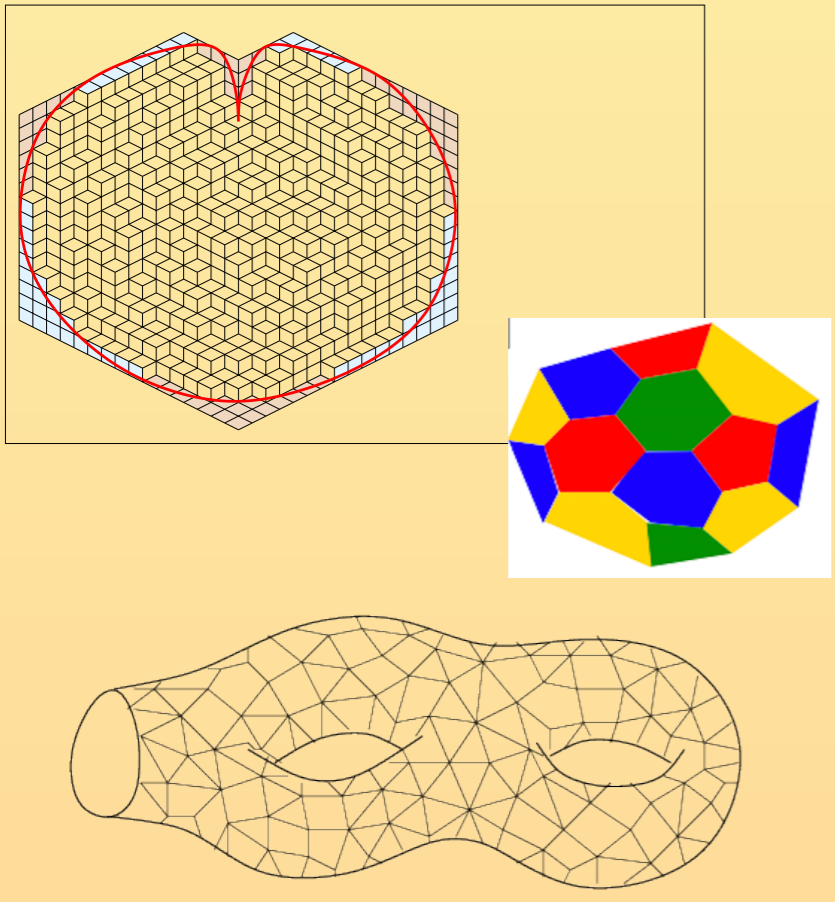


Géométries aléatoires dans le miroir de la géométrie algébrique



B. Eynard,
IPHT CEA Saclay, CRM Montréal

Plan



- **Compter les surfaces: une motivation en physique:**
la théorie des cordes

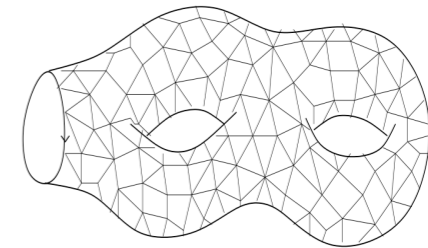
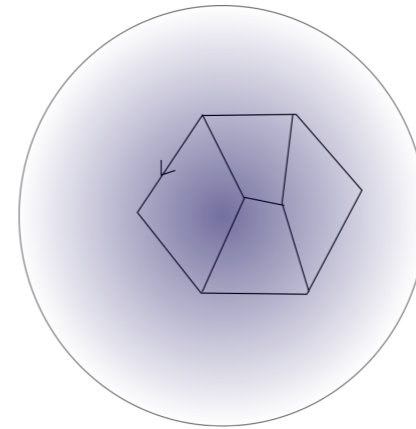
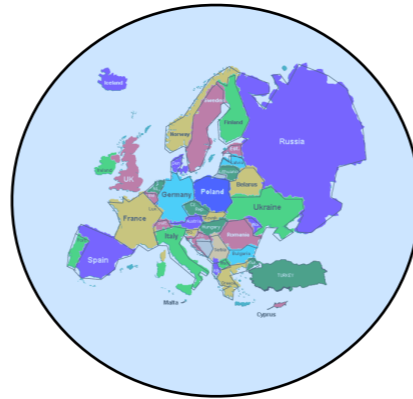


- **Introduction à la symétrie miroir**



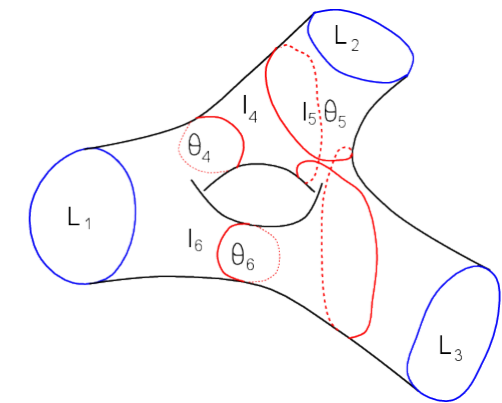
- **Compter les « cartes »**

- planaires
- Autres topologies



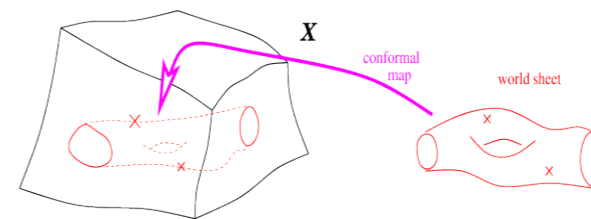
- **Compter les surfaces hyperboliques** (*courbure négative*)

- la récurrence de **Maryam Mirzakhani**



- **Compter les surfaces immergées**

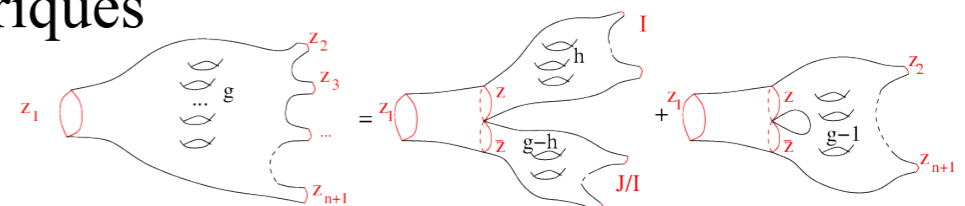
- les invariants de **Gromov-Witten**



- **Récurrence topologique et symétrie miroir**



- du comptage des surfaces
aux invariants géométriques des courbes algébriques



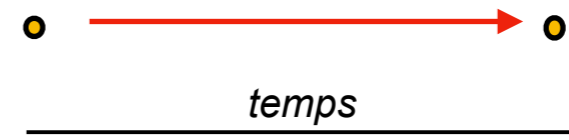
Une motivation : *Théorie des cordes*

(en très résumé)



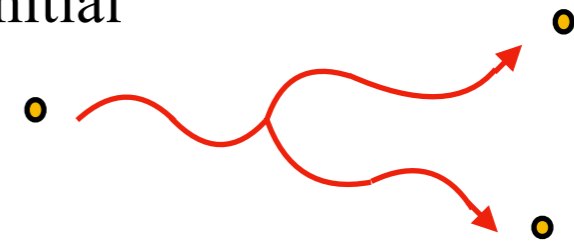
Physique classique: particule = •

- Les particules sont des points qui se déplacent le long de trajectoires (lignes droites)



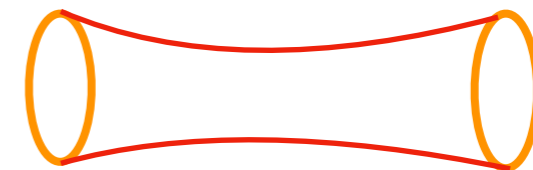
Physique quantique:

- **Probabiliste** : probabilité (*complexe*) d'un état final, sachant l'état initial = « *nombre* » de trajectoires possibles



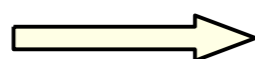
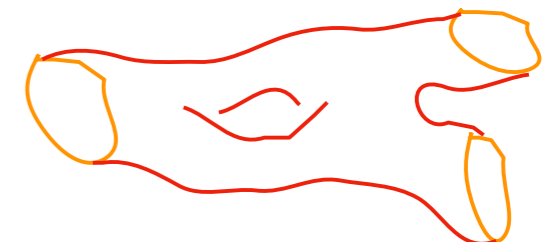
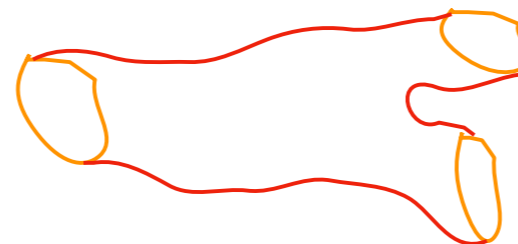
Cordes classiques:

- Les particules sont des « cordes » qui se déplacent le long de trajectoires (surfaces)



Cordes quantiques:

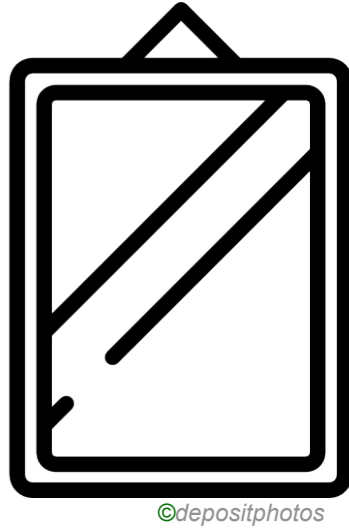
- **Probabiliste** : loi de probabilité sur les surfaces, sachant les bords = « *nombre* » de surfaces possibles



Nécessité de « compter » (*mesurer*) des ensembles de surfaces



Symétrie Miroir



Notion de symétrie Miroir en théorie des cordes:

[Candelas, de la Ossa, Green, Parkes (1991)]

« A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory »

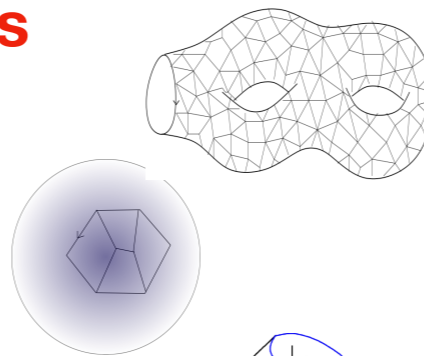
Notion de symétrie Miroir en mathématiques: [Kontsevich (1994)]

• A-side :

- Géométrie énumérative
- Compter les configurations (surfaces)
- Compter = mesure (probabilité)
- Géométrie aléatoire
- Mesure sur un espace de surfaces

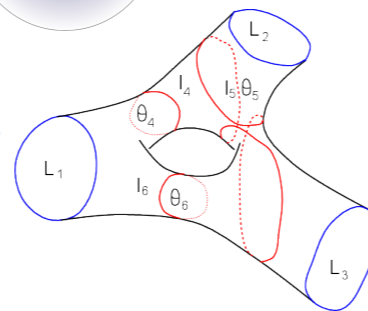
• Compter les « cartes »

- Planaires
- Genre g



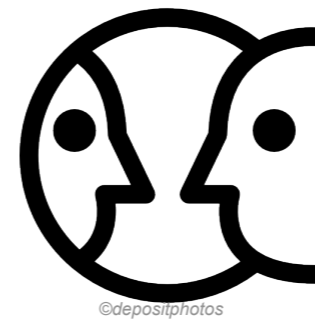
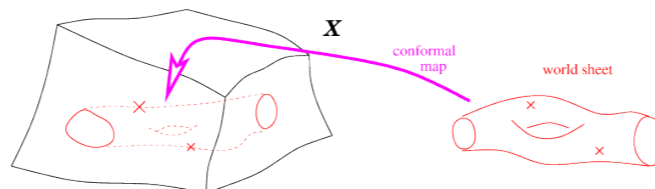
• Compter les surfaces hyperboliques

Maryam Mirzakhani's recursion



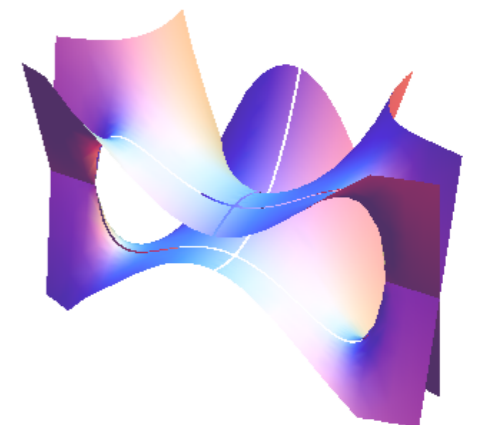
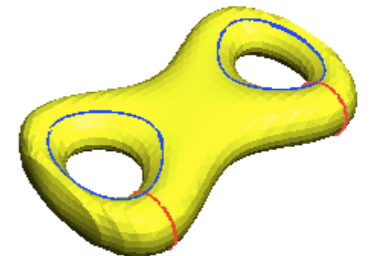
• Compter les immersions de surfaces dans un espace cible

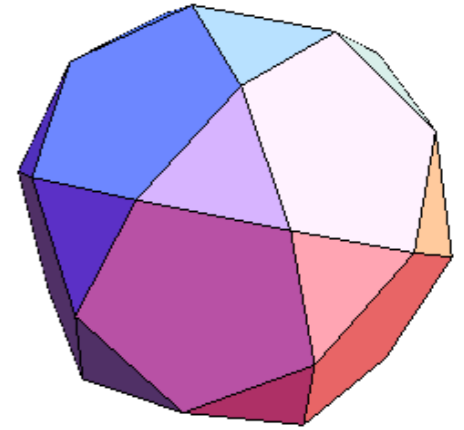
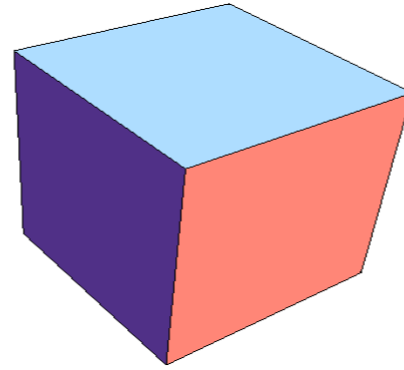
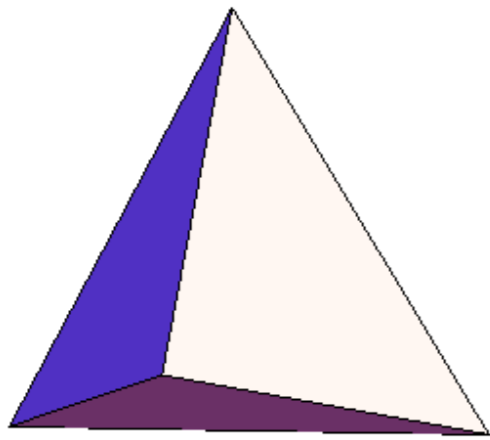
invariants de Gromov-Witten



• B-side :

- Géométrie algébrique
- Analyse complexe
- Périodes
- Résidus

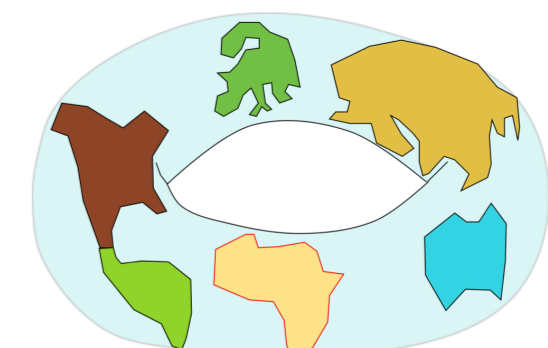
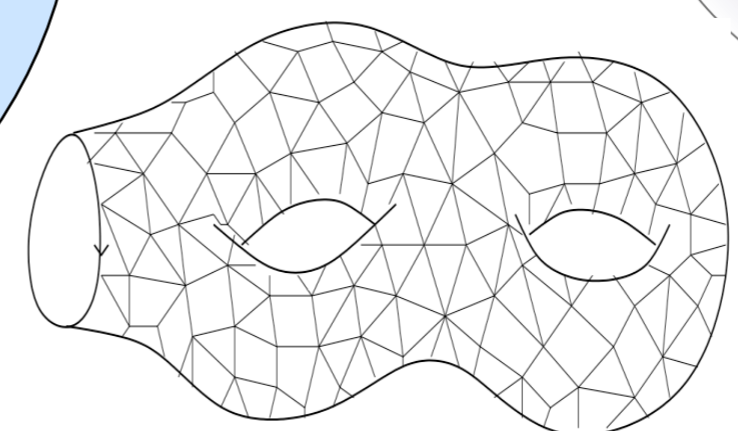
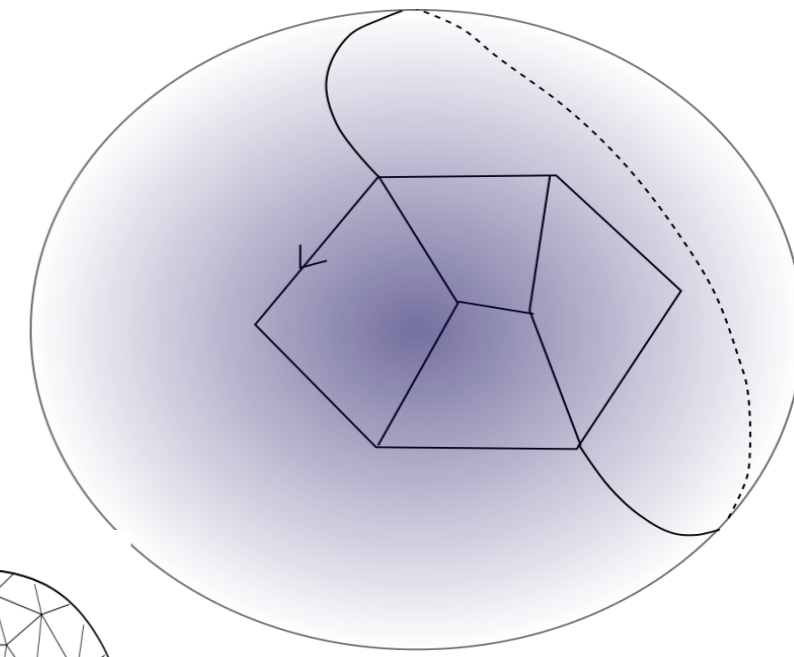




Compter les surfaces discrètes



Aussi appelées « cartes »



Surfaces faites de quadrangles



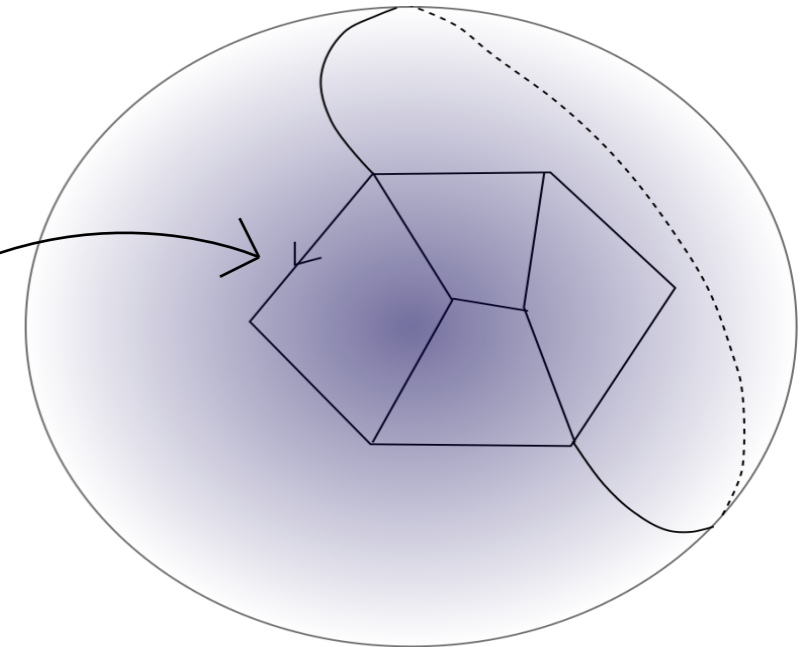
W. Tutte's formula [1960's] :

Nombre de quadrangulations « enracinées » de la sphère avec k faces =

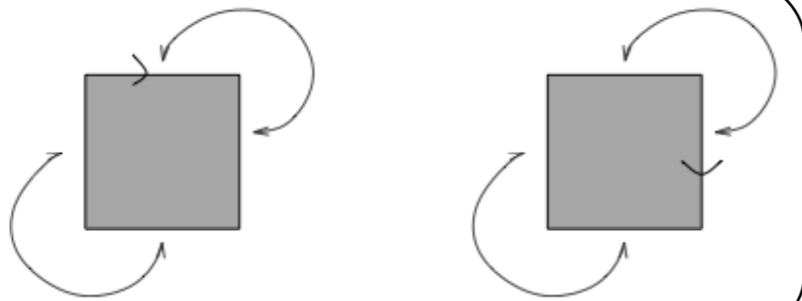
2, 9, 54, 378, 2916, 24057, ...

$$\frac{2 \cdot 3^k (2k)!}{k!(k+2)!}$$

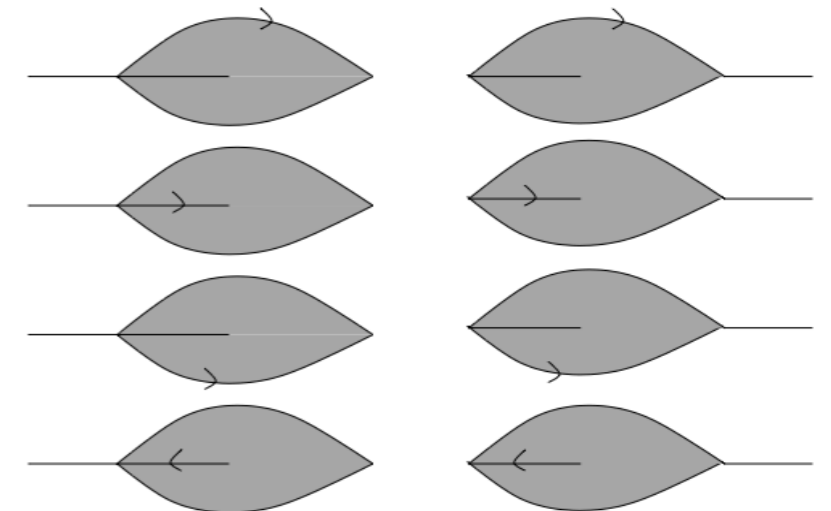
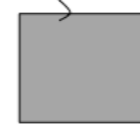
1 arête racine



2 =



9 =



• On fait une **série génératrice**:

$$2t + 9t^2 + 54t^3 + 378t^4 + 2916t^5 + \dots$$

→ Donne :

$$\sum_{k=1}^{\infty} t^k \frac{2 \cdot 3^k (2k)!}{k!(k+2)!} = 4t \frac{1 + 3r\sqrt{1-12t}}{(1+r)^3 (1 + \sqrt{1-12t})^3}$$

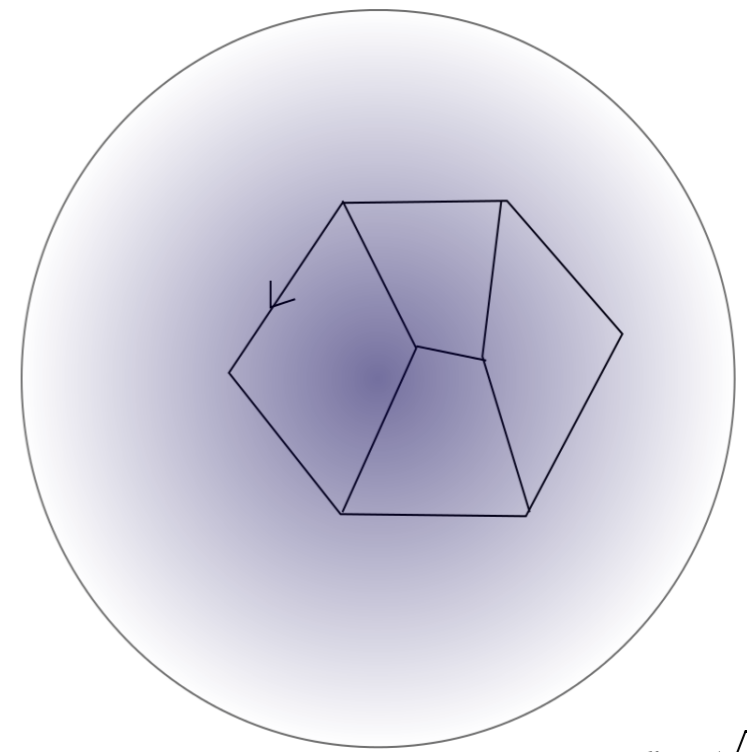
où : $r = \sqrt{1-12t}$

Application miroir $t \mapsto r$



Surfaces faites de quadrangles

Nombre de **quadrangulations**
de la **sphère** avec
1 face marquée enracinée de longueur **2l**
et **k faces non-marquées** :



2l=6

k=4

→ 225

$$r = \sqrt{1 - 12t}$$

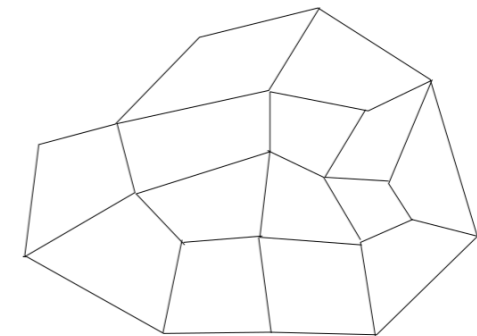
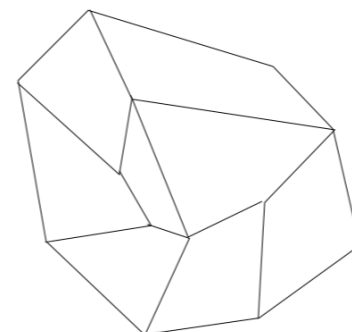
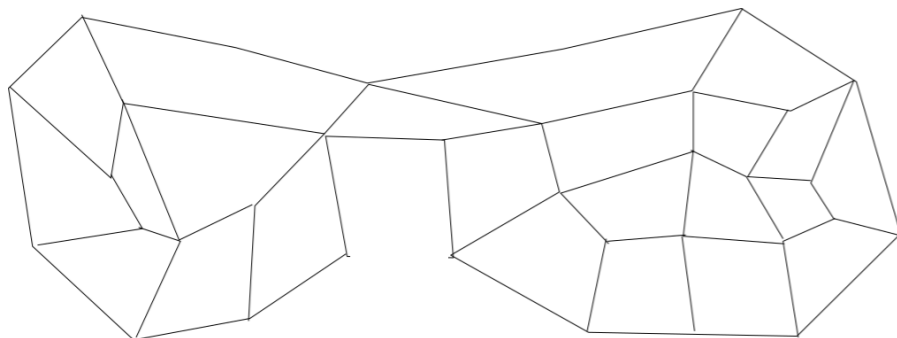
$$\sum_l \sum_k \frac{t^k}{z^{2l+1}} 3^k \frac{2l!}{l!(l-1)!} \frac{(2k+l-1)!}{k!(l+k+1)!} = \frac{1}{2} \left(z - tz^3 + \left(tz^2 - \frac{2+r}{3} \right) \sqrt{z^2 - \frac{8}{1+r}} \right)$$

$$= W_{0,1}(z; t)$$

Preuve:

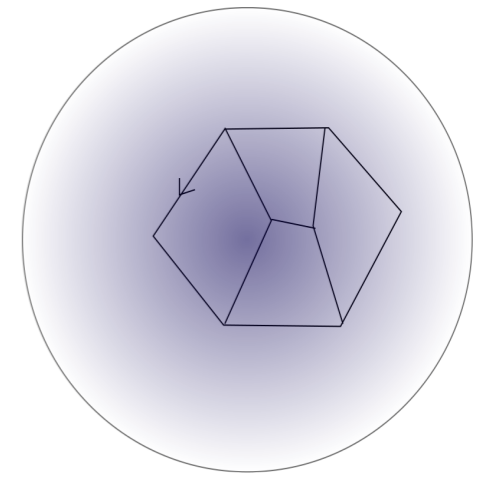
Par récurrence: effacer l'arête marquée

$$zW_{0,1}(z; t) = tz^3W_{0,1}(z; t) + W_{0,1}(z; t)^2 - tz^2 + 1 - \frac{1+r-2r^2}{9(1+r)}$$

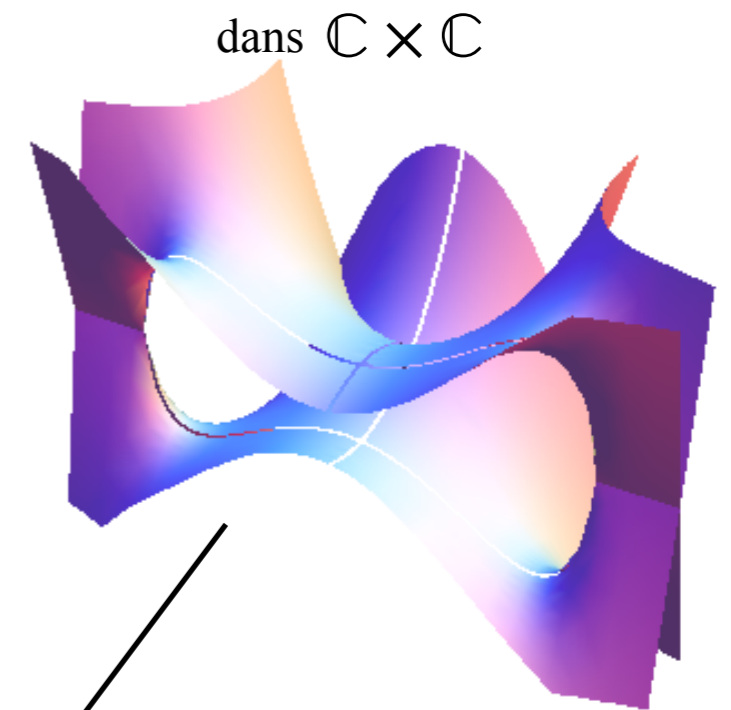
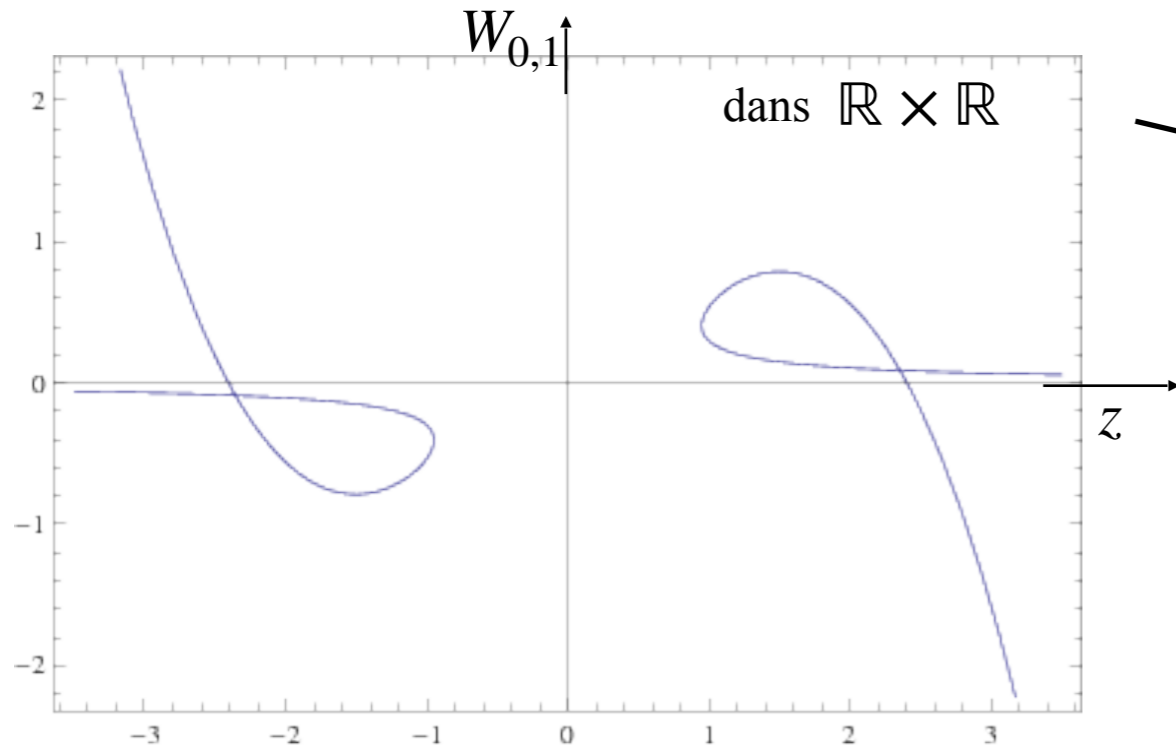


Courbe spectrale

$$W_{0,1}(z; t) = \frac{1}{2} \left(z - tz^3 - \left(tz^2 - \frac{2+r}{3} \right) \sqrt{z^2 - \frac{8}{1+r}} \right)$$



$$r = \sqrt{1 - 12t}$$



Bijection conforme avec la sphère de Riemann

Paramétrisation

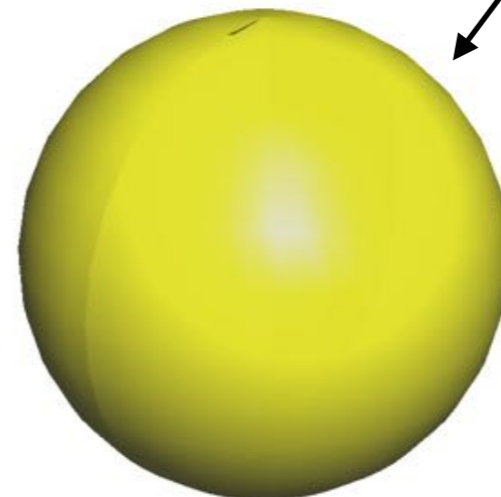
(variable x)

$x \in \text{Sphère } \mathbb{C}P^1$

$$z = \sqrt{\frac{2}{1+r}} \left(x + \frac{1}{x} \right)$$

$$W_{0,1}(z; t) = \sqrt{\frac{1+r}{2}} \left(\frac{1}{x} - \frac{1-r}{3(1+r)} \frac{1}{x^3} \right)$$

bijection conforme



Sphère de Riemann = $\mathbb{C} \cup \{\infty\}$
 = $\mathbb{C}P^1$



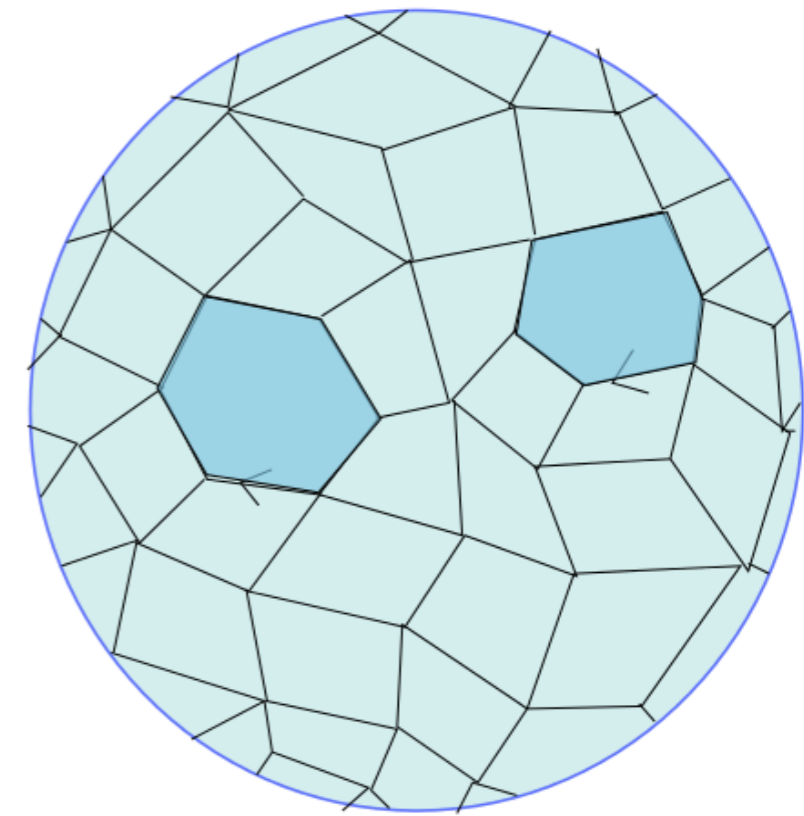
Surfaces faites de quadrangles

Sphère

2 faces marquées enracinées

2 faces enracinées de longueurs
et k faces non-marquées

$$l_1 = \frac{a+b}{2}, l_2 = \frac{a-b}{2}$$



$$\sum_{l_1, l_2} \sum_k \frac{t^k}{z_1^{l_1+1} z_2^{l_2+1}} \quad 3^k a \frac{(2k+a-1)!}{k!(k+a)!} \sum_{j=0}^{a-b} \frac{(a-b-2j)(a+b)!(a-b)!}{j!(b+j)!(a-j)!(a-b-j)!}$$

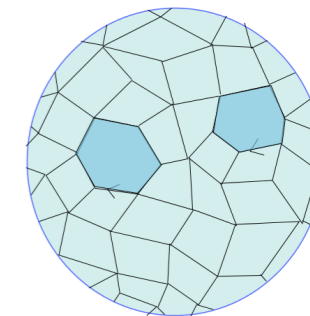
$$= W_{0,2}(z_1, z_2; t) = \frac{1}{2(z_1 - z_2)^2} \left(-1 + \frac{z_1 z_2 - \frac{8}{1+r}}{\sqrt{z_1^2 - \frac{8}{1+r}} \sqrt{z_2^2 - \frac{8}{1+r}}} \right)$$

$$r = \sqrt{1-12t}$$



Forme fondamentale

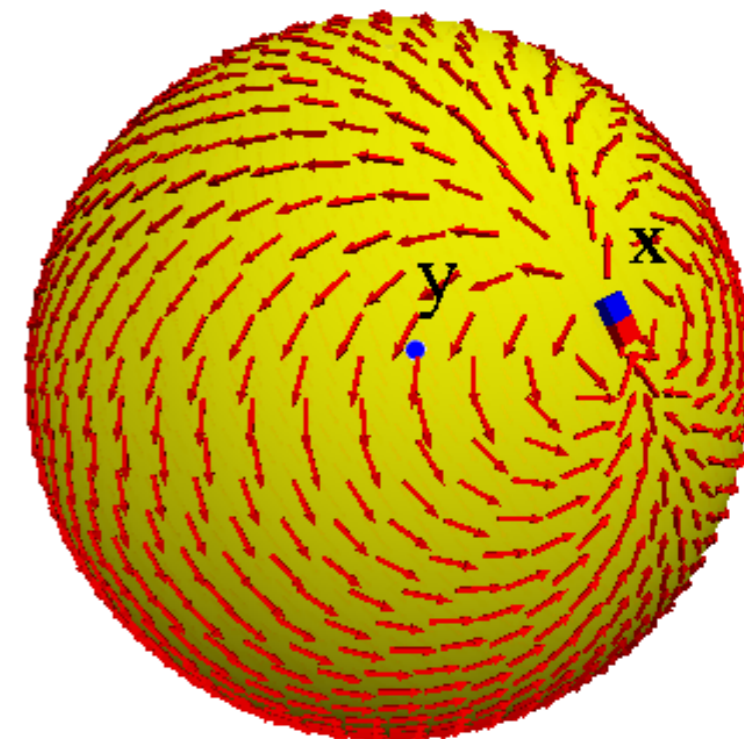
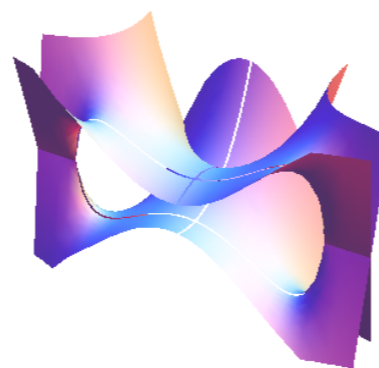
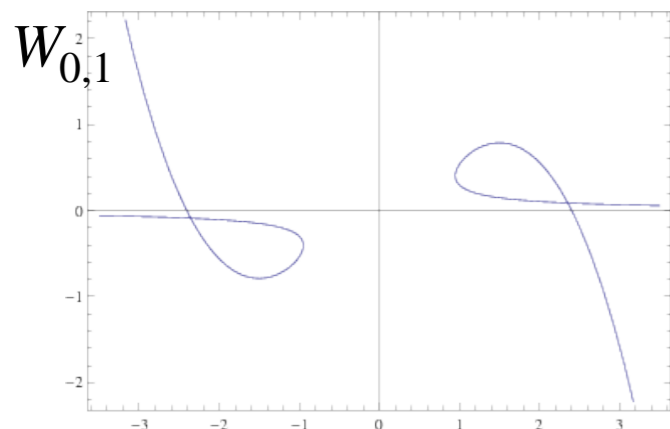
de la courbe spectrale



$$r = \sqrt{1 - 12t}$$

$$W_{0,2}(z_1, z_2; t) = \frac{1}{2(z_1 - z_2)^2} \left(1 + \frac{z_1 z_2 - \frac{8}{1+r}}{\sqrt{z_1^2 - \frac{8}{1+r}} \sqrt{z_2^2 - \frac{8}{1+r}}} \right) - \frac{1}{(z_1 - z_2)^2}$$

$$= \frac{1}{(x_1 - x_2)^2} \frac{dx_1 dx_2}{dz_1 dz_2} - \frac{1}{(z_1 - z_2)^2} = \frac{d}{dz_1} \frac{d}{dz_2} \ln \frac{x_1 - x_2}{z_1 - z_2}$$



Paramétrisation

(variable x)

$$z = \sqrt{\frac{2}{1+r}} \left(x + \frac{1}{x} \right)$$

$$W_{0,1}(z; t) = \sqrt{\frac{1+r}{2}} \left(\frac{1}{x} - \frac{1-r}{3(1+r)} \frac{1}{x^3} \right)$$

$$z_1 = \sqrt{\frac{2}{1+r}} \left(x_1 + \frac{1}{x_1} \right)$$

$$z_2 = \sqrt{\frac{2}{1+r}} \left(x_2 + \frac{1}{x_2} \right)$$

$W_{0,2}$ = fonction de "Green" sur la Courbe Spectrale

2^{nde} forme fondamentale

Noyau de Bergmann

$$W_{0,2}(z_1, z_2; t) dz_1 dz_2 = \frac{dx_1 dx_2}{(x_1 - x_2)^2} - \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

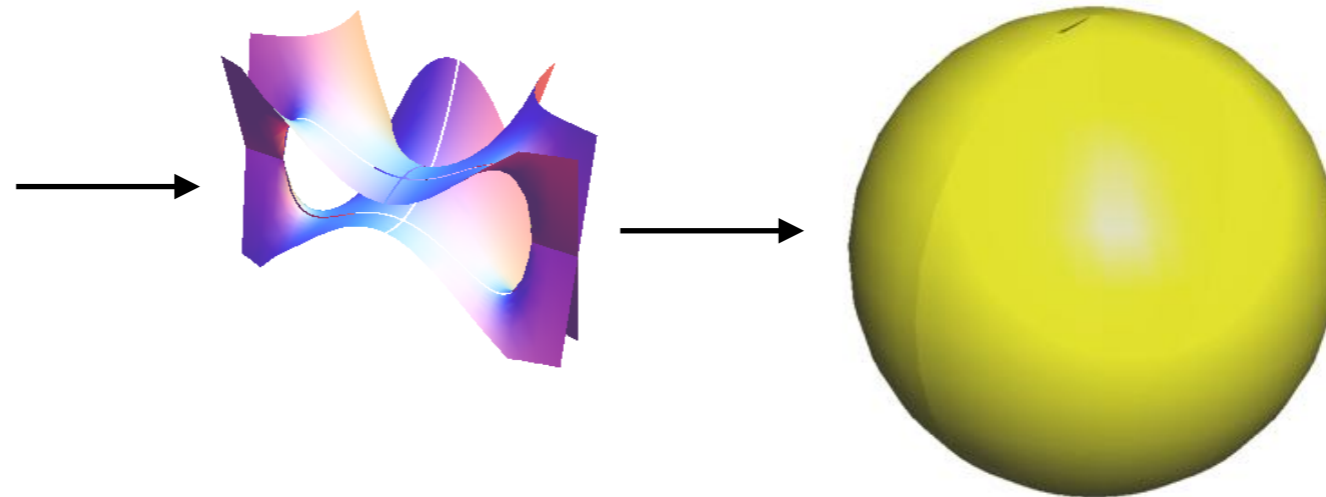
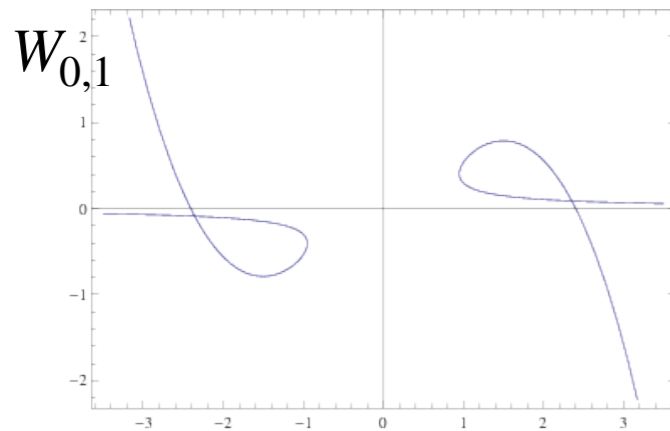
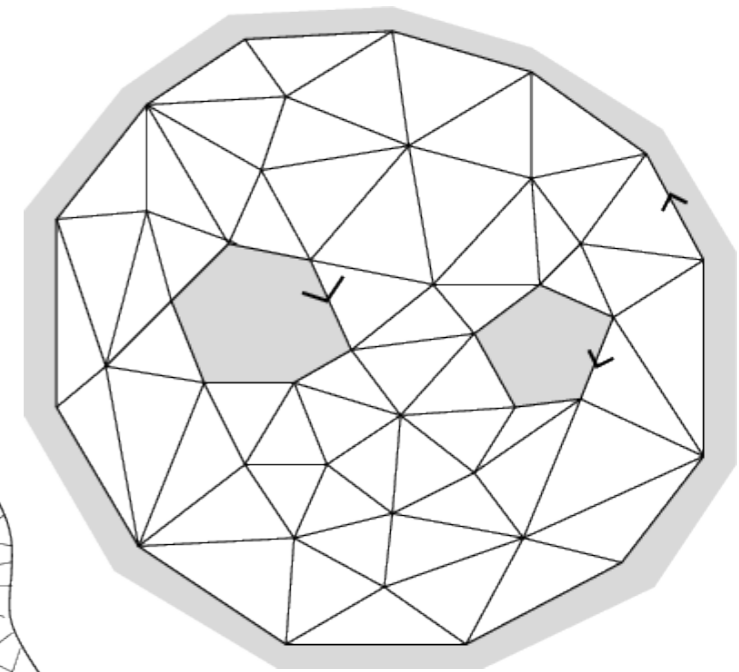
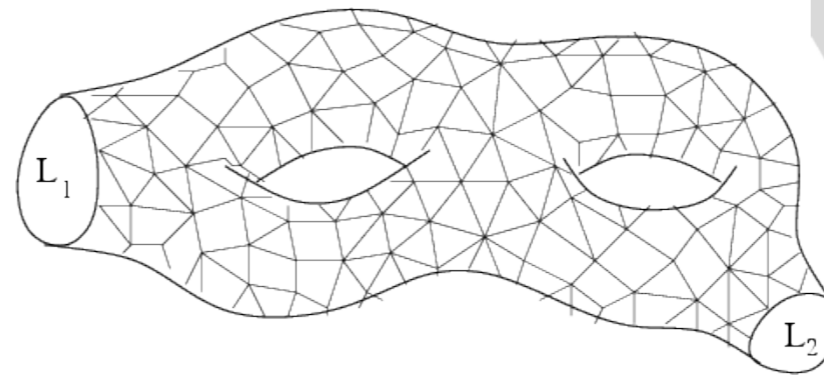


Surfaces faites de quadrangles

Genre g

n faces marquées enracinées

$$W_{g,n}(z_1, \dots, z_n; t)$$



Question:

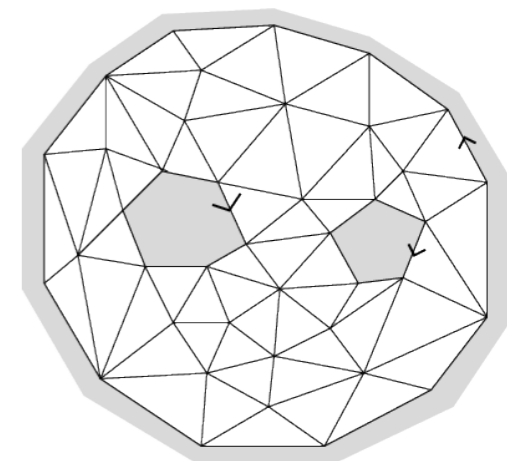
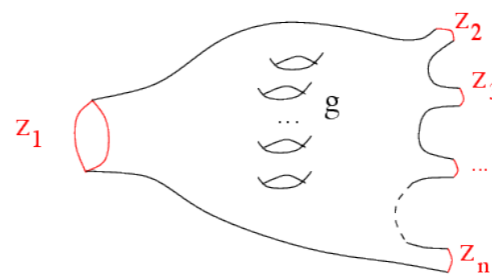
- Comment calculer $W_{g,n}(z_1, \dots, z_n; t)$?
- Est-elle une fonction algébrique ?
- A-t-elle une interprétation géométrique sur la courbe spectrale ?



Récurrance topologique

Genre g n faces marquées enracinées

$$W_{g,n}(z_1, \dots, z_n; t) =$$



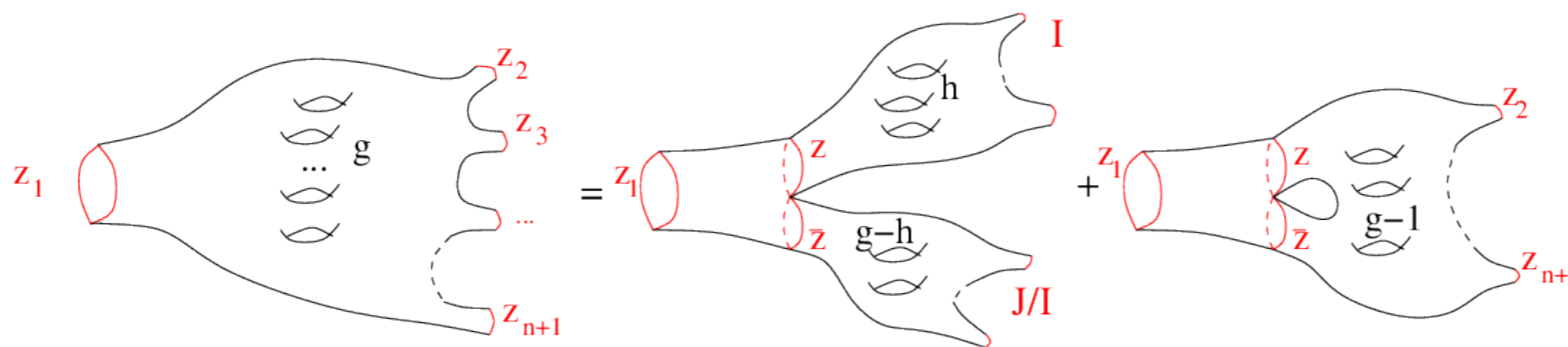
Théorème:

pour tout $n \geq 1$ et $(g, n) \neq (0, 1), (0, 2)$ on a:

[E2004]

$$W_{g,n}(z_1, \dots, z_n) \sqrt{z_1^2 - \frac{8}{1+r}} = \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{dz}{z - z_1} \frac{1}{tz^2 - \frac{2+r}{3}} \left[W_{g-1, n+1}(z, z, z_2, \dots, z_n) \right. \\ \left. + \sum_{\substack{\text{no discs} \\ g_1 + g_2 = g \\ I_1 \uplus I_2 = \{z_2, \dots, z_n\}}} W_{g_1, 1 + \#I_1}(z, I_1) W_{g_2, 1 + \#I_2}(z, I_2) \right]$$

$\mathcal{C} =$ contour entourant $\pm \sqrt{\frac{8}{1+r}}$



$$K(z_1, z) = \frac{1}{z_1 - z} \frac{1}{tz^2 - \frac{2+r}{3}} \frac{1}{\sqrt{z_1^2 - \frac{8}{1+r}}}$$

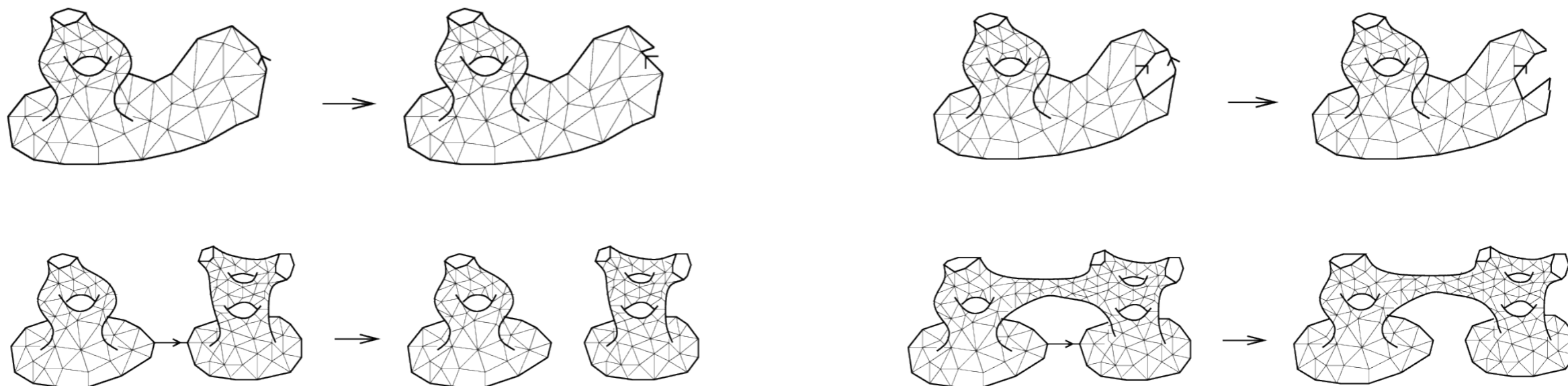


$$W_{0,2}(z_1, z_2) =$$



Preuve

à la Tutte: on enlève récursivement l'arête marquée de la face 1

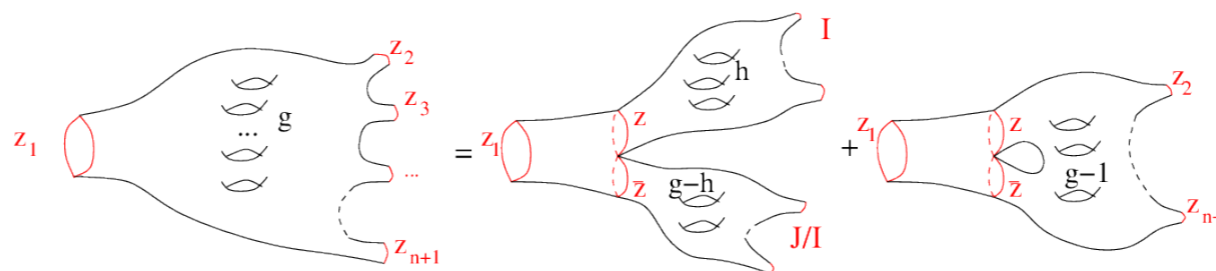


Théorème:

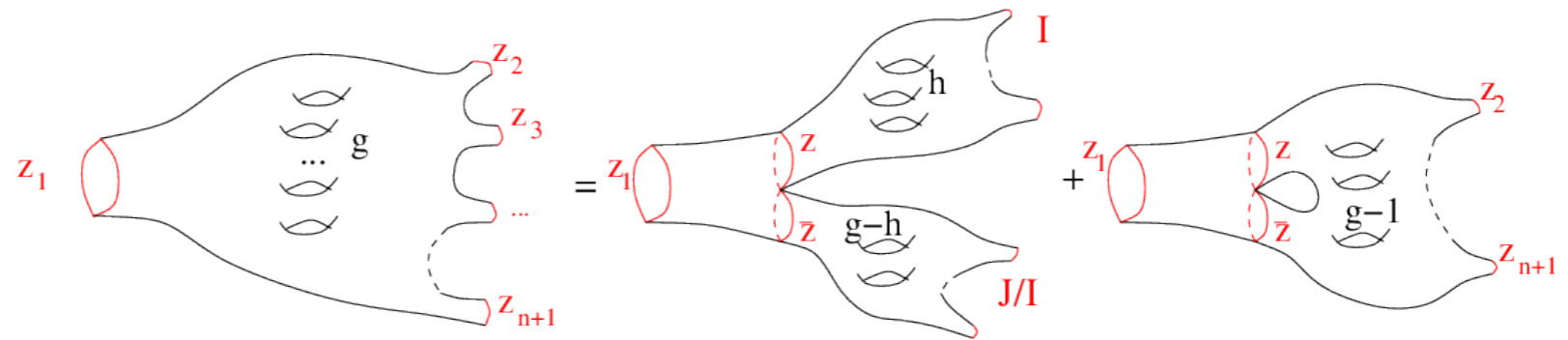
pour tout $n \geq 1$ et $(g, n) \neq (0, 1), (0, 2)$ on a:

$$W_{g,n}(z_1, \dots, z_n) \sqrt{z_1^2 - \frac{8}{1+r}} = \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{dz}{z - z_1} \frac{1}{tz^2 - \frac{2+r}{3}} \left[W_{g-1, n+1}(z, z, z_2, \dots, z_n) \right. \\ \left. + \sum_{\substack{\text{no discs} \\ g_1+g_2=g \\ I_1 \uplus I_2 = \{z_2, \dots, z_n\}}} W_{g_1, 1+\#I_1}(z, I_1) W_{g_2, 1+\#I_2}(z, I_2) \right]$$

$\mathcal{C} =$ contour entourant $\pm \sqrt{\frac{8}{1+r}}$



Examples:



$$W_{0,3}(z_1, z_2, z_3) = 2 \text{ (diagram of a tube with three ends)} \\ = 2 \oint_{\mathcal{E}} K(z_1, z) W_{0,2}(z, z_2) W_{0,2}(z, z_3)$$

$$W_{1,1}(z_1) = \text{(diagram of a tube with one end and a loop)} \\ = \oint_{\mathcal{E}} K(z_1, z) W_{0,2}(z, z)$$

$$W_{0,4}(z_1, z_2, z_3, z_4) = \text{(diagram of a tube with four ends)} + \text{sym} \dots$$

$$= \oint_{\mathcal{E}} K(z_1, z) [W_{0,3}(z, z_2, z_3) W_{0,2}(z, z_4) + \text{sym}] \\ = \oint_{\mathcal{E}} K(z_1, z) \oint_{\mathcal{E}} K(z, z') [W_{0,2}(z', z_2) W_{0,2}(z', z_3) W_{0,2}(z, z_4) + \text{sym}]$$

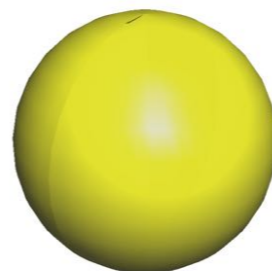
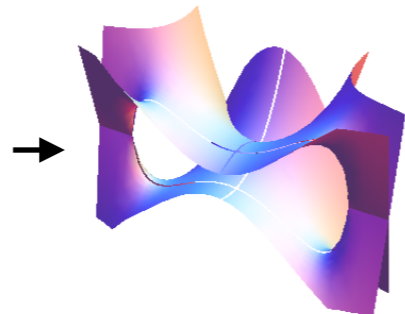
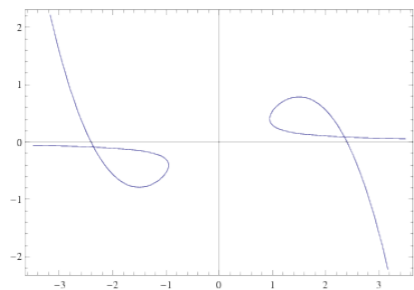
$$W_{1,2}(z_1, z_2) = \text{(diagram 1)} + \text{(diagram 2)}$$

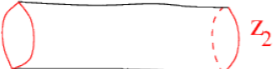


Récurrance topologique et géométrie de la courbe spectrale

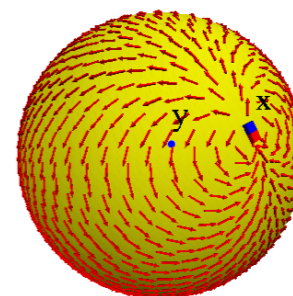
- $W_{0,1}(z)$ = courbe spectrale


(= calculée par les équations de Tutte)



- $W_{0,2}(z_1, z_2) =$ 

= forme fondamentale sur la courbe spectrale



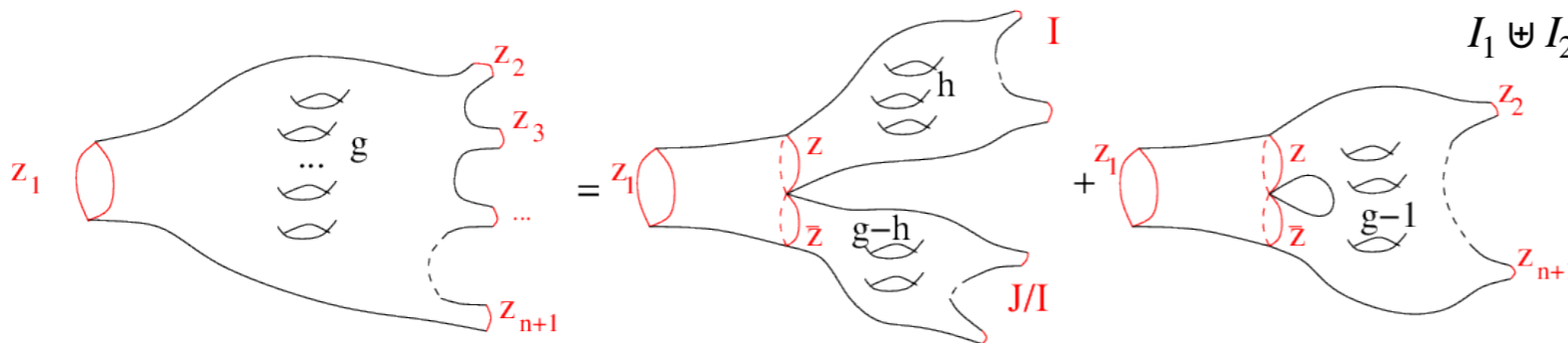
- $K(z_1, z) =$  $= \frac{\int_a^z W_{0,2}(z_1, z') dz'}{W_{0,1}(z^+) - W_{0,1}(z^-)}$

$$a = \sqrt{\frac{8}{1+r}}$$

- Par récurrence

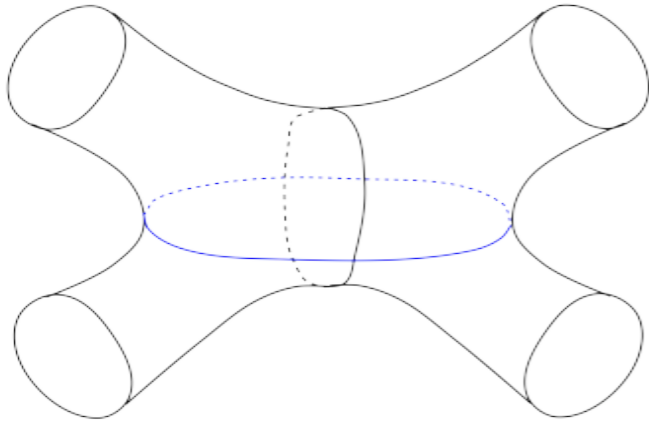
$$W_{g,n}(z_1, \dots, z_n) = \frac{1}{2i\pi} \oint_{\mathcal{C}} dz K(z_1, z) \left[W_{g-1, n+1}(z, \bar{z}, z_2, \dots, z_n) + \sum_{\substack{\text{no discs} \\ g_1+g_2=g}} W_{g_1, 1+\#I_1}(z, I_1) W_{g_2, 1+\#I_2}(z, I_2) \right]$$

$I_1 \uplus I_2 = \{z_2, \dots, z_n\}$

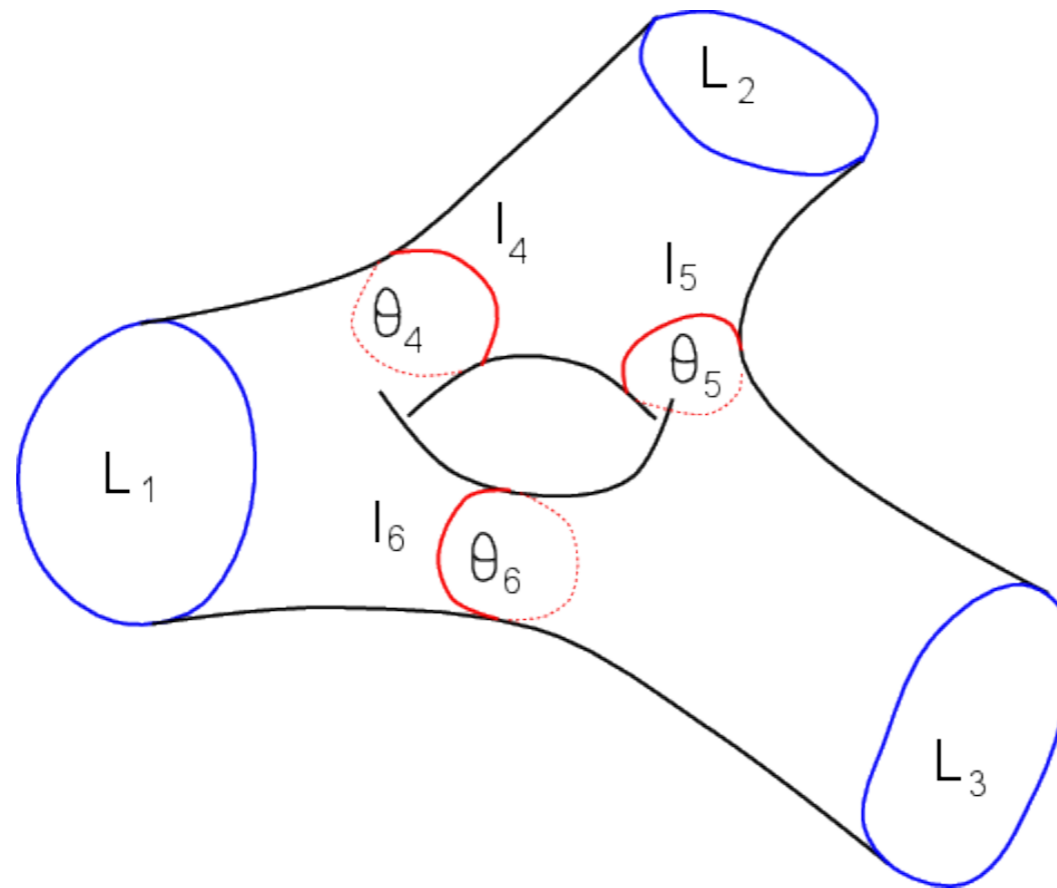


$W_{g,n}(z_1, \dots, z_n)$ = champs de vecteur multipolaires sur la surface





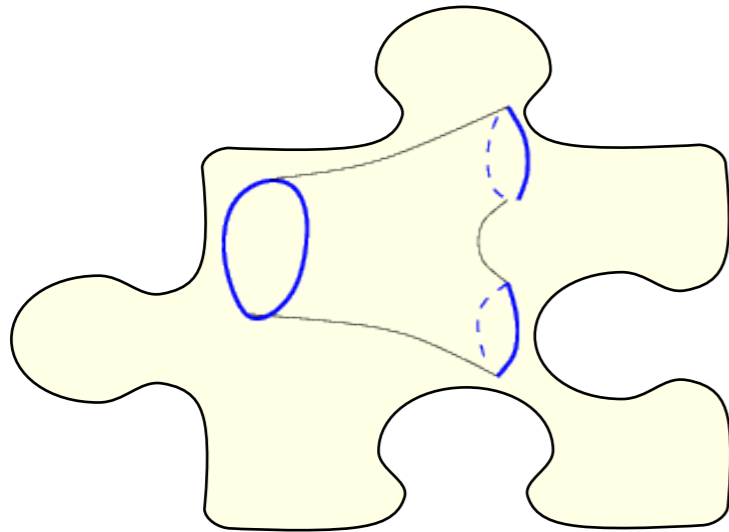
II. Compter les surfaces hyperboliques



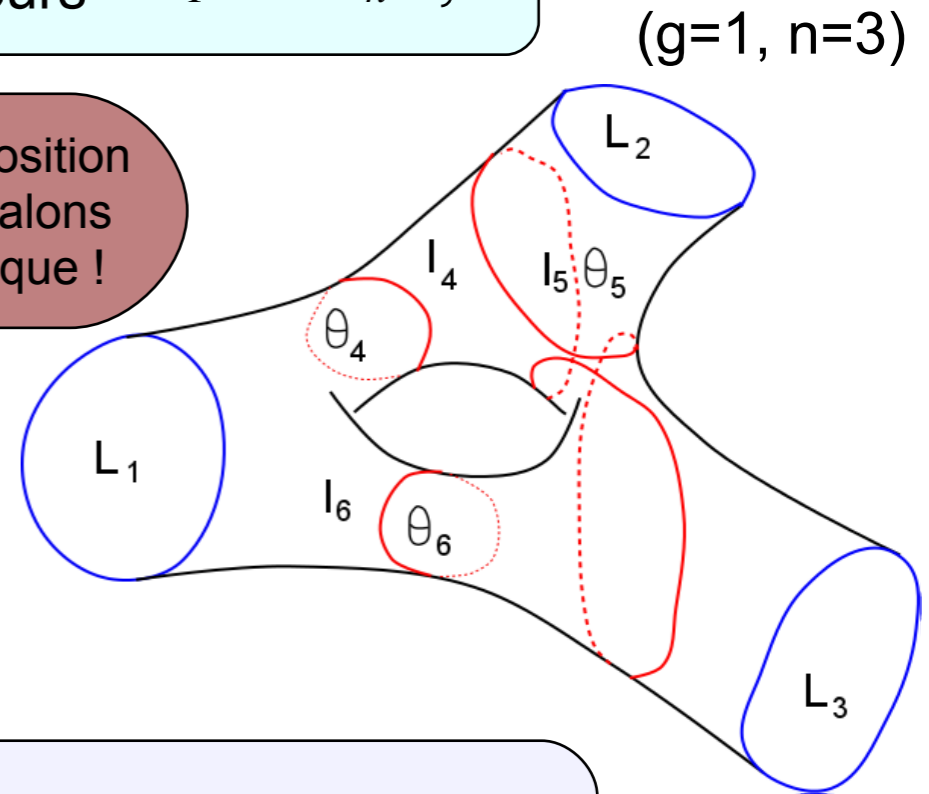
Compter les surfaces hyperboliques

$$M_{g,n}(L_1, \dots, L_n) = \left\{ \begin{array}{l} \text{surfaces de courbure constante } R=-1 \\ \text{avec } n \text{ bords géodesiques de longueurs } L_1, \dots, L_n \end{array} \right\}$$

- Decomposition en «pantalons»



Decomposition
En pantalons
Pas unique !



(g=1, n=3)



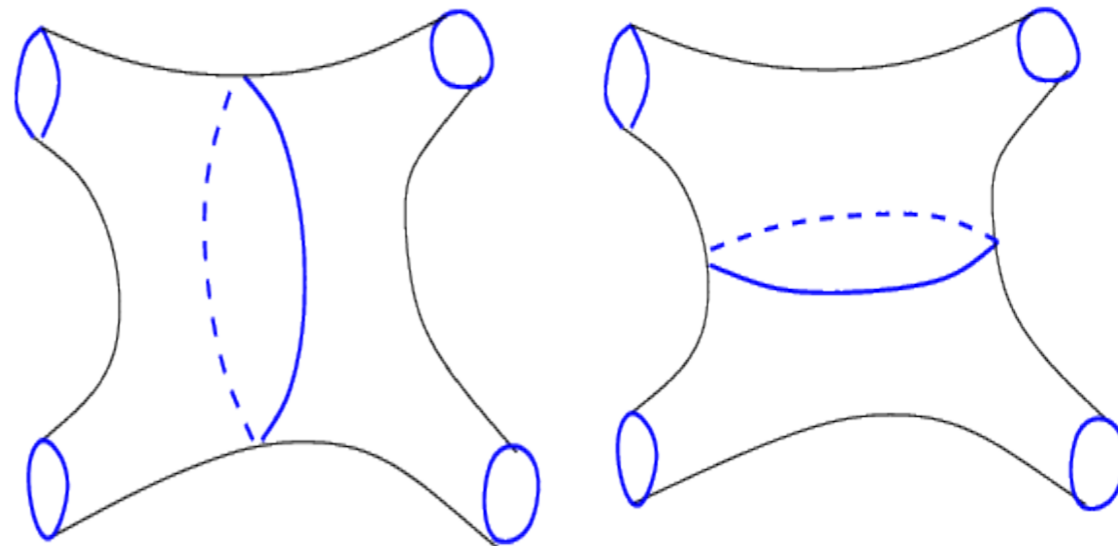
- Volume (*Weil - Petersson*) :

$$V_{g,n}(L_1, \dots, L_n) = \int \prod_e dl_e \wedge d\theta_e$$

$$V_{0,3}(L_1, L_2, L_3) = 1$$

$$V_{1,1}(L_1) = \frac{1}{24}(2\pi^2 + L_1^2)$$

$$V_{0,4}(L_1, \dots, L_4) = 2\pi^2 + \frac{1}{2} \sum_{i=1}^4 L_i^2$$



(g=0, n=4)



Comptage récursif

- transformée de Laplace des volumes

$$W_{g,n}(z_1, \dots, z_n) = \int_0^\infty \dots \int_0^\infty L_1 dL_1 e^{-z_1 L_1} \dots L_n dL_n e^{-z_n L_n} V_{g,n}(L_1, \dots, L_n)$$

Théorème : [Mirzakhani 2004]

Laplace transformed [EO 2006]

$$W_{g,n}(z_1, \dots, z_n) = \text{Res}_{z \rightarrow 0} K(z_1, z) \left[W_{g-1, n+1}(z, \bar{z}, z_2, \dots, z_n) + \sum_{\substack{\text{no discs} \\ g_1 + g_2 = g \\ I_1 \uplus I_2 = \{z_2, \dots, z_n\}}} W_{g_1, 1 + \#I_1}(z, I_1) W_{g_2, 1 + \#I_2}(z, I_2) \right]$$

$$K(z_1, z) = \frac{\pi}{(z_1^2 - z^2) \sin(2\pi z)}$$

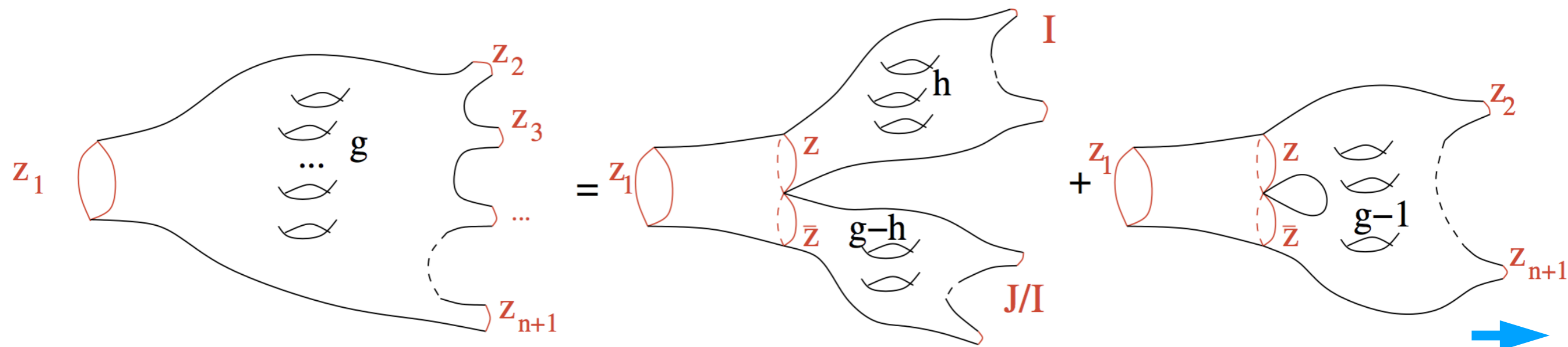
$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

Interprétation :

- On compte les surfaces de genre g , à n bords
- par récurrence sur leur **caractéristique d'Euler**

$$\chi = 2 - 2g - n$$

Récurrance Topologique



Comptage récursif

Théorème : [Mirzakhani 2004 + EO 2006]

$$W_{g,n}(z_1, \dots, z_n) = \text{Res}_{z \rightarrow 0} K(z_1, z) \left[W_{g-1, n+1}(z, -z, z_2, \dots, z_n) + \sum W_{g_1, |I_1|+1}(z, I_1) W_{g_2, |I_2|+1}(-z, I_2) \right]$$

$$K(z_1, z) = \frac{\pi}{(z_1^2 - z^2) \sin(2\pi z)}$$

$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

• Exemple
$$W_{1,1}(z_1) = \int L_1 dL_1 e^{-z_1 L_1} \left(\frac{L_1^2}{24} + \frac{2\pi^2}{24} \right) = \frac{3}{24z_1^4} + \frac{2\pi^2}{24z_1^2}$$

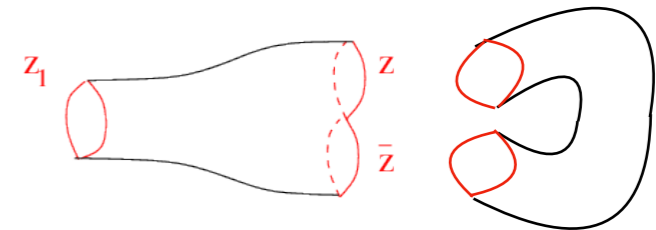
$$W_{1,1}(z_1) = \text{Res}_{z \rightarrow 0} K(z_1, z) [W_{0,2}(z, -z)]$$

$$= \text{Res}_{z \rightarrow 0} \frac{\pi}{(z_1^2 - z^2) \sin(2\pi z)} \frac{1}{4z^2}$$

$$= \frac{1}{8z_1^2} \text{Res}_{z \rightarrow 0} \frac{1}{z^3} \left(1 - \frac{z^2}{z_1^2} \right)^{-1} \left(1 - \frac{1}{6}(2\pi z)^2 + \dots \right)^{-1}$$

$$= \frac{1}{8z_1^2} \text{Res}_{z \rightarrow 0} \frac{1}{z^3} \left(1 + \frac{z^2}{z_1^2} + \frac{1}{6}(2\pi z)^2 + \dots O(z^4) \right)$$

$$= \frac{1}{8z_1^2} \left(\frac{1}{z_1^2} + \frac{1}{6}(2\pi)^2 \right) = \frac{3}{24z_1^4} + \frac{2\pi^2}{24z_1^2}$$



Miroir de la géométrie hyperbolique

Théorème : [Mirzakhani 2004]

Laplace transformed [EO 2006]

$$W_{g,n}(z_1, \dots, z_n) = \text{Res}_{z \rightarrow 0} K(z_1, z) \left[W_{g-1, n+1}(z, \bar{z}, z_2, \dots, z_n) + \sum_{\substack{\text{no discs} \\ g_1+g_2=g \\ I_1 \uplus I_2 = \{z_2, \dots, z_n\}}} W_{g_1, 1+\#I_1}(z, I_1) W_{g_2, 1+\#I_2}(z, I_2) \right]$$

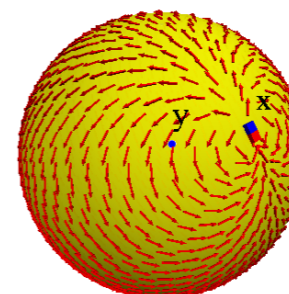
$$K(z_1, z) = \frac{\pi}{(z_1^2 - z^2) \sin(2\pi z)}$$

$$W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

• **Miroir:**

$$W_{0,2}(z_1, z_2) = z_1 \text{ (cylinder) } z_2$$

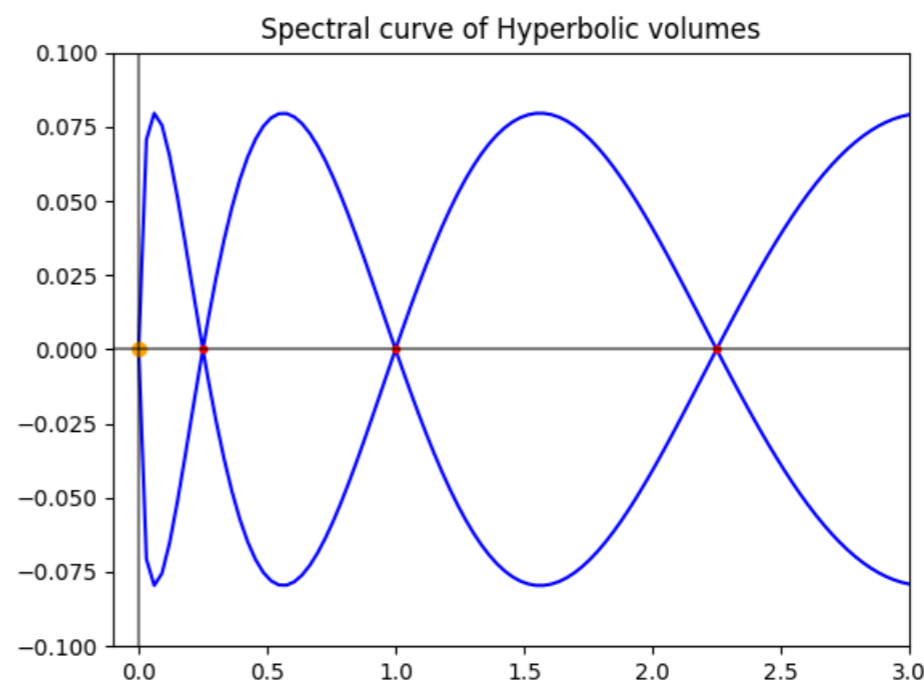
= forme fondamentale sur la courbe spectrale



$$K(z_1, z) = z_1 \text{ (cylinder) } z \bar{z} = \frac{\int_a^z W_{0,2}(z_1, z') dz'}{W_{0,1}(z^+) - W_{0,1}(z^-)}$$

Courbe spectrale

$$W_{0,1}(z) = \frac{1}{4\pi} \sin 2\pi z = \frac{1}{4\pi} \sin 2\pi\sqrt{x}$$

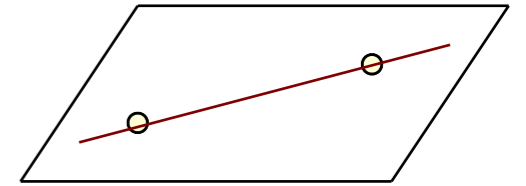


III. *Invariants de Gromov-Witten*

Compter les courbes (complexes) dans le plan

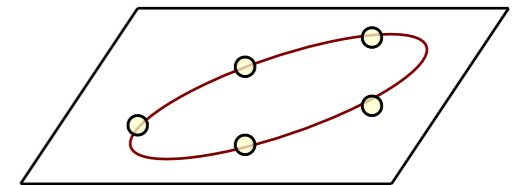
• Combien de droites (complexes) par 2 points dans $\mathbb{C}P^2$? \Rightarrow

1



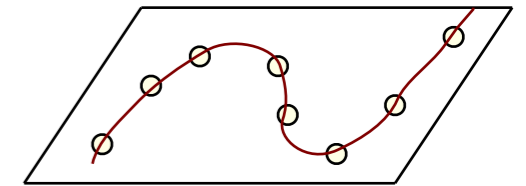
• Combien de coniques par 5 points dans $\mathbb{C}P^2$? \Rightarrow

1



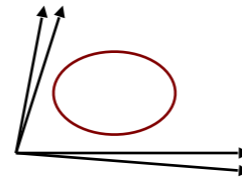
• Combien de courbes rationnelles de degré 3 par 8 points dans $\mathbb{C}P^2$? \Rightarrow

12



• Combien de courbes rationnelles de degré d par $3d-1$ points dans $\mathbb{C}P^2$? \Rightarrow

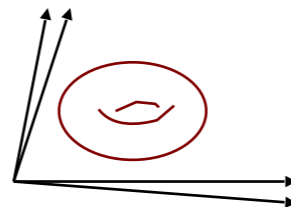
$N_0(\mathbb{C}P^2, d)$



$= 1, 1, 12, 620, 87304, \dots$

• Combien de courbes elliptiques de degré d par $3d$ points dans $\mathbb{C}P^2$? \Rightarrow

$N_1(\mathbb{C}P^2, d)$



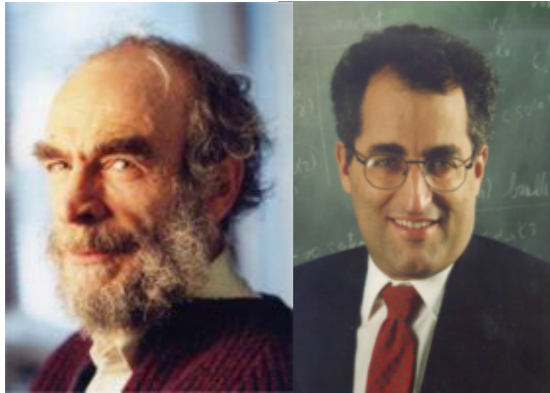
$= 0, 0, 1, 225, \dots$

• Combien de courbes de genre g et de degré d par $3d+g-1$ points dans $\mathbb{C}P^2$?

\Rightarrow nombres de **Gromov-Witten** $N_g(\mathbb{C}P^2, d)$



Compter les surfaces immergées dans un espace cible



- Combien de courbes de genre g et de degré d holomorphiquement immergées dans X ?

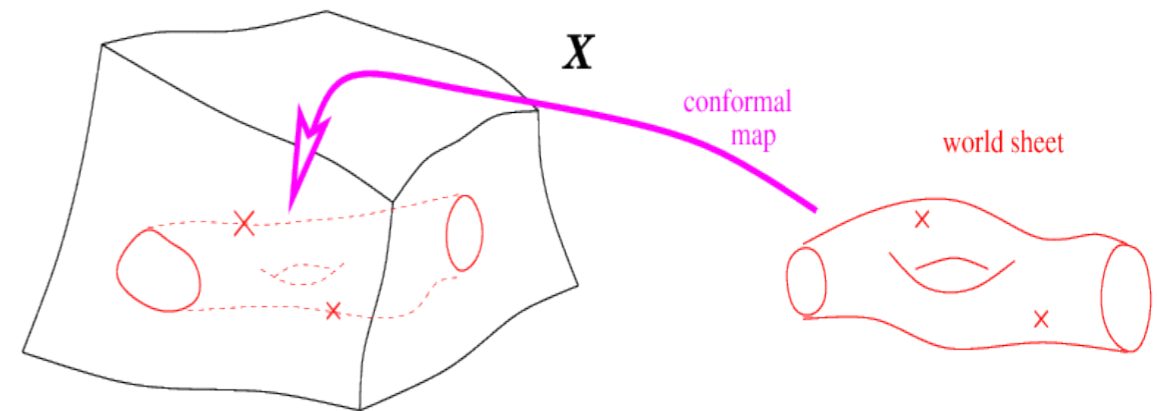
$$d \in H_2(X)$$

$$\mathcal{M}_g(X, d) = \{(\Sigma, f) \mid \text{genre}(\Sigma) = g, f : \Sigma \rightarrow X, \deg f(\Sigma) = d\}_{/isom}$$

nombres de Gromov-Witten : $N_g(X, d) = \int_{\mathcal{M}_g(X, d)} 1$

Et avec n bords (degrés d_1, \dots, d_n):

$$N_{g,n}(X, d, d_1, \dots, d_n)$$



Séries génératrices:

$$W_{g,0}(t) = \sum_d t^d N_g(X, d)$$

$$W_{g,n}(z_1, \dots, z_n; t) = \sum_{d, d_1, \dots, d_n} t^d e^{d_1 z_1} \dots e^{d_n z_n} N_{g,n}(X, d, d_1, \dots, d_n)$$

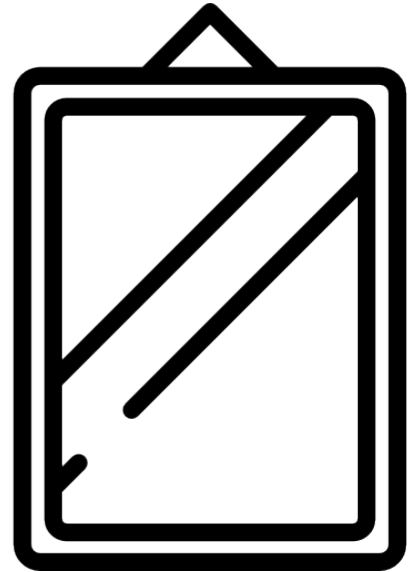
Comment les calculer ?

Symétrie miroir

Récurrence topologique



Compter les surfaces immergées dans un espace cible



Notion de symétrie Miroir:

[Candelas, de la Ossa, Green, Parkes (1991)]

« *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory* »

[Hori, Vafa (2000)] « *Mirror Symmetry* »

Théorie des cordes de type A (géométrie aléatoire) Théorie des cordes de type B (périodes)
 Calabi-Yau X \leftrightarrow Calabi-Yau \hat{X}

[Kontsevich (1994)]

introduces : conjecture of « Homological Mirror symmetry »

Conjecture BKMP(2008)

Il existe une fonction algébrique

$$X = \text{Calabi-Yau Torique} \\ W_{0,1}(z) = \text{miroir de } X = \hat{X} \quad \text{☺}$$

Telle que Récurrence topologique

$$\implies W_{g,n}(z_1, \dots, z_n; t) = \text{Invariants de Gromov-Witten de } X$$

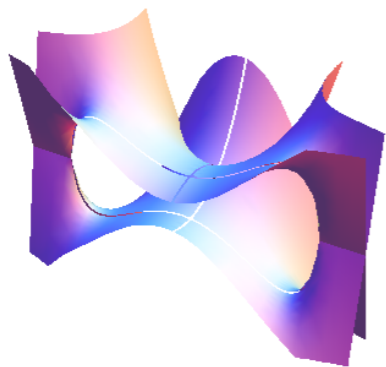
Théorème [EO2012], [BLZ2014]

Example $X = \mathbb{C}^3$ $\hat{X} = \{(z, w) \mid e^z + e^w = 1\}$ $W_{0,1}(z) = \ln(1 - e^z)$



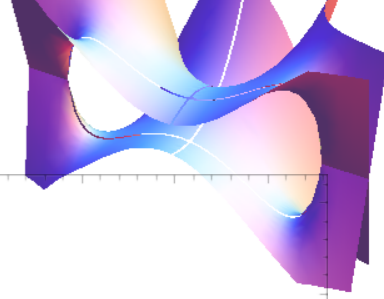
IV. Récurrence Topologique

*Nouveaux Invariants
pour les courbes algébriques*



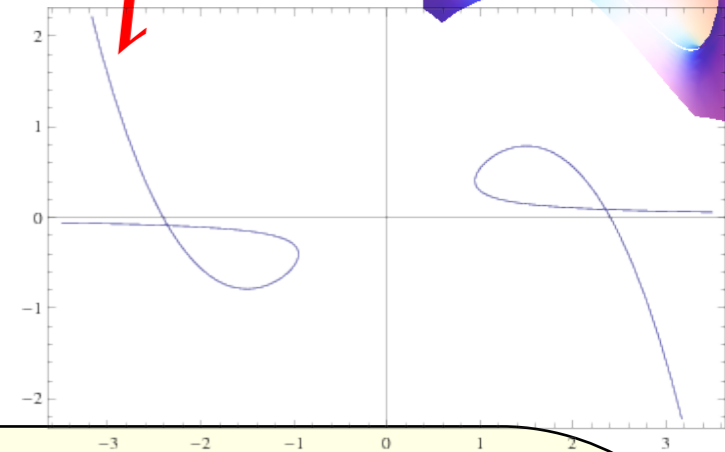
$$\left\{ W_{g,n} \right\}_{\substack{g=0,\dots,\infty \\ n=0,\dots,\infty}}$$

Invariants d'une courbe algébrique



On choisit arbitrairement une

$$\text{Courbe spectrale } S = \{(z, w) \mid P(z, w) = 0\}$$



Definition: les *invariants* de S

(Définition universelle: pas de paramètre)

$$W_{0,1}(z) = w \, dz$$

$$W_{0,2}(z_1, z_2) = \text{Forme fondamentale}$$

Et récursivement *(Récurrence Topologique)*

$$W_{g,n}(z_1, \dots, z_n) = K(z_1, z) * \left[W_{g-1, n+1}(z, \bar{z}, z_2, \dots, z_n) + \sum W_{g_1, |I_1|+1}(z, I_1) W_{g_2, |I_2|+1}(\bar{z}, I_2) \right]$$

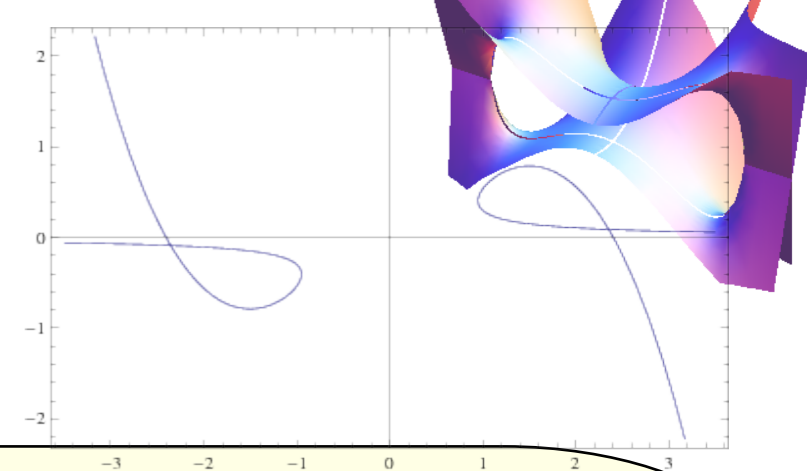
$$\text{avec } K(z_1, z) = \frac{1}{2} \frac{\int W_{0,2}(z_1, z')}{W_{0,1}(z) - W_{0,1}(\bar{z})}$$

Nombreuses (belles) propriétés:

- Invariants symplectiques
- Formes quasi-modulaires
- Intégrabilité \rightarrow • τ - fonction
- Virasoro • et plein d'autres ... etc ...



Symétrie miroir inverse



Courbe spectrale $S = \{(z, w) \mid P(z, w) = 0\}$

Invariants de S

$$W_{0,1}(z) = w \, dz \quad W_{0,2}(z_1, z_2) = \text{Forme fondamentale}$$

Et récursivement $W_{g,n}(z_1, \dots, z_n)$

Théorème [E2011]: Courbe spectrale \rightarrow Mesure de comptage de surfaces

Transformée de Laplace:

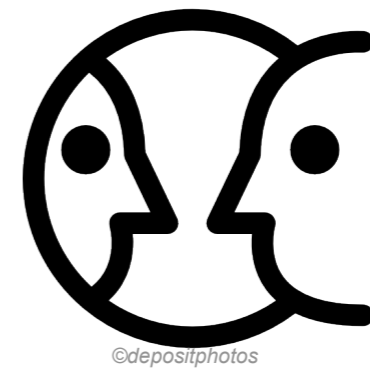
$$f(u) = \frac{u^{3/2}}{\sqrt{2\pi}} \int_0^\infty e^{-z(u-u_0)} W_{0,1}(z) \, dz \quad \ln f(u) = - \sum_{k=0}^\infty f_k u^{-k}$$

$$\hat{B}(u, v) = \frac{\sqrt{uv}}{\pi} \int_0^\infty \int_0^\infty dz_1 dz_2 e^{-z_1(u-u_0)} e^{-z_2(u-u_0)} W_{0,2}(z_1, z_2) = \sum_{k,l} B_{k,l} u^{-k} v^{-l}$$

$$\tilde{B}(u, z) = \frac{\sqrt{u}}{\sqrt{\pi}} \int_0^\infty dz_2 e^{-z_2(u-u_0)} W_{0,2}(z, z_2) = \sum_k \xi_k(z) u^{-k}$$

Alors $W_{g,n}(z_1, \dots, z_n) = \int_{\overline{\mathcal{M}}_{g,n}} e^{\sum_{k=0}^\infty f_k \kappa_k + \frac{1}{2} \sum_{k,l} B_{k,l} \delta_* \psi^k \psi^l} \prod_{i=1}^n \left(\sum_{d=0}^\infty \psi_i^d \xi_d(z_i) \right)$

Conclusion



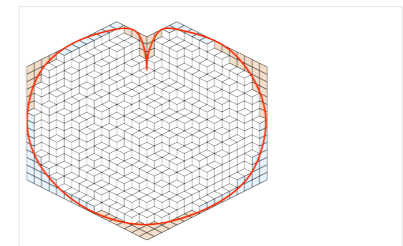
- La **symétrie miroir** est très puissante
- transforme un problème de **géométrie aléatoire** (*mesure sur un espace de surfaces*) en un problème de **géométrie algébrique** (*périodes et résidus*) sur une courbe algébrique (*surface de Riemann*).
- Pour les **cartes**: si on sait compter les « disques », alors on a une **courbe spectrale**
cartes quadrangulées -> miroir = $zW_{0,1}(z; t) = tz^3W_{0,1}(z; t) + W_{0,1}(z; t)^2 - tz^2 + 1 - \frac{1+r-2r^2}{9(1+r)}$
- Pour les **surfaces hyperboliques** miroir = $W_{0,1}(z) = \frac{1}{4\pi} \sin 2\pi\sqrt{z}$
- **Inversement**: étant donnée une courbe spectrale arbitraire (un miroir) :
la récurrence topologique définit ses « **invariants** »
la transformée de Laplace définit un **espace de surfaces et une mesure**

• Autres exemples:

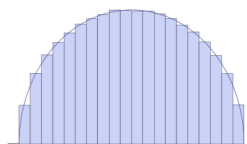
noeud -> miroir = A-polynome



Crystal -> miroir = courbe arctique

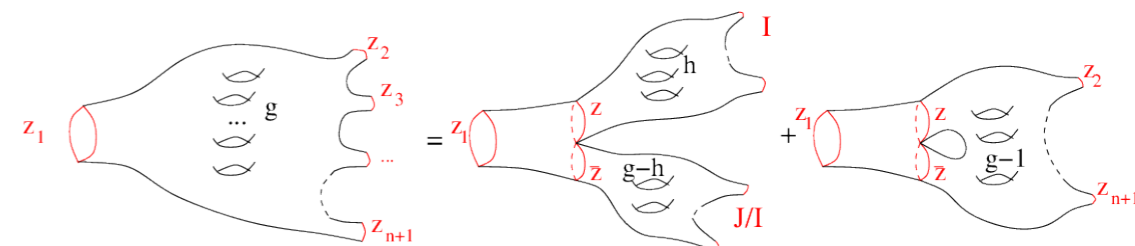
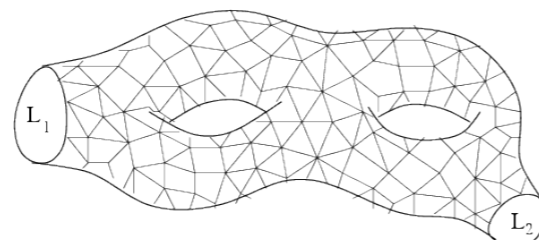


Matrices aléatoires -> miroir = densité de valeurs propres



- Ces invariants ont plein de **belles propriétés** mathématiques
systèmes intégrables, théories conformes, théorie des noeuds, matrices aléatoires, ...

• **A suivre ...**



Merci de votre attention

