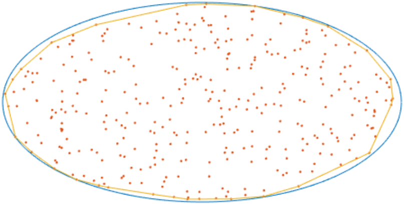
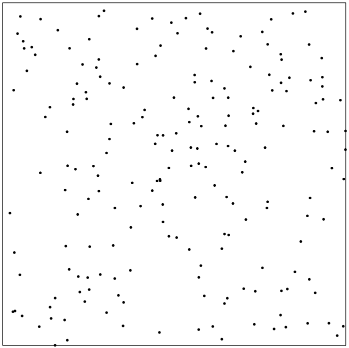
Close-up on random convex geometry



Pierre Calka

January 28, 2025
Collège de France
Géométries aléatoires et applications

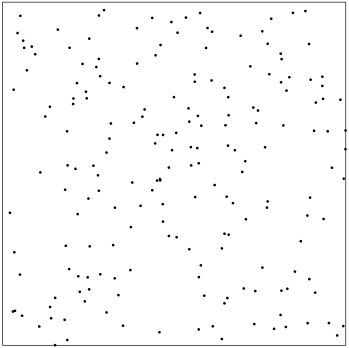
Generate random points in a compact window W



Binomial point process

Fix $n \geq 1$, generate n independent points uniformly distributed in W

Generate random points in a compact window W

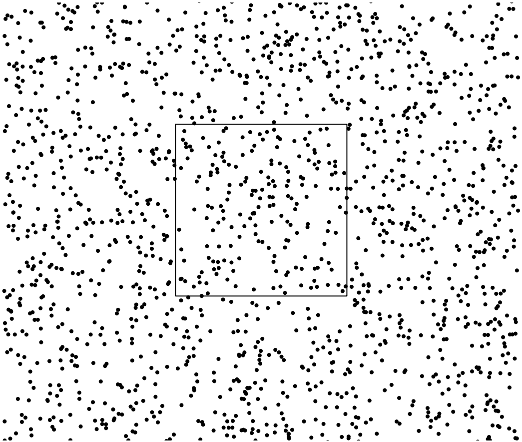


Poisson point process in W

Fix $\lambda \geq 1$, generate Pois(λ) independent points uniformly distributed in W

When W has unit volume, $\lambda :=$ intensity

Generate random points in $\mathbb{R}^d$



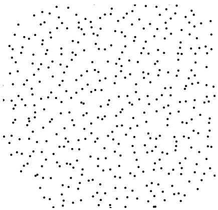
Homogeneous Poisson point process in $\mathbb{R}^d$

Same intersection with W and independence between disjoint windows

Generate random points in $\mathbb{R}^d$

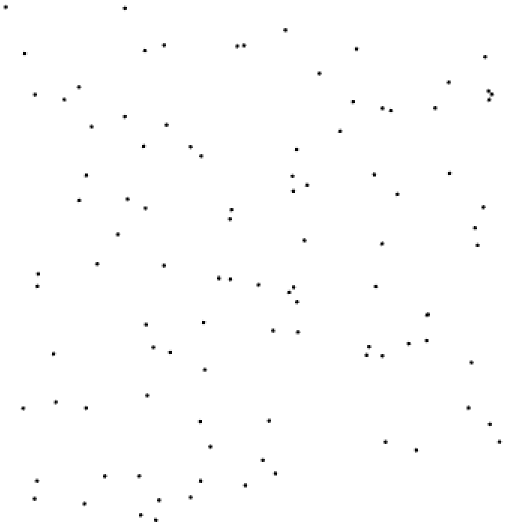
Gaussian Poisson point process

Matérn cluster point process

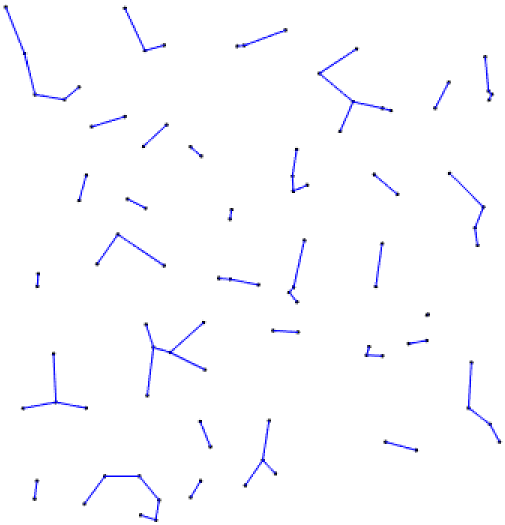


Ginibre determinantal point process

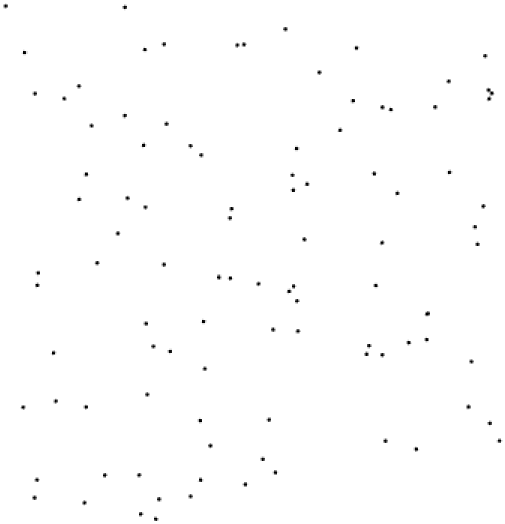
Make a deterministic geometric construction



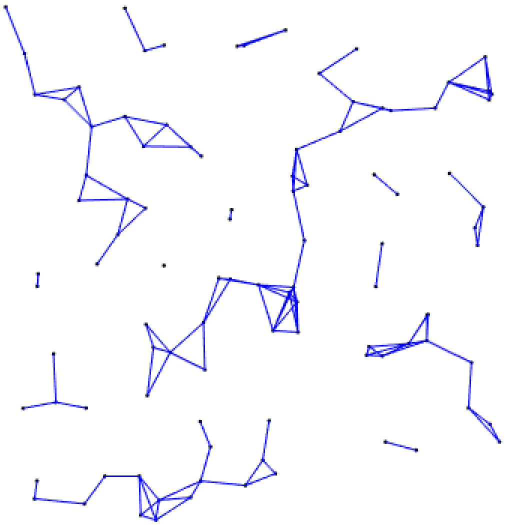
Nearest-neighbor graph



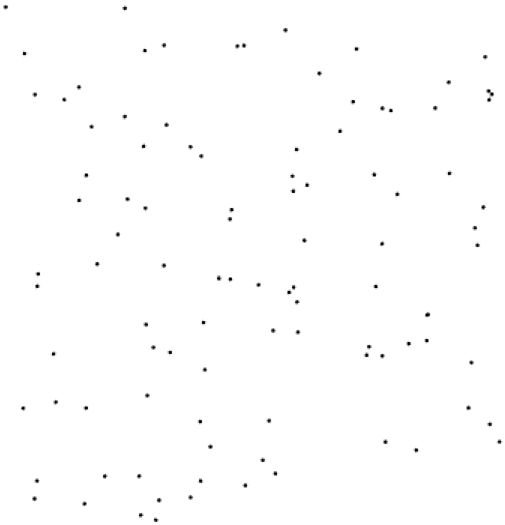
Make a deterministic geometric construction



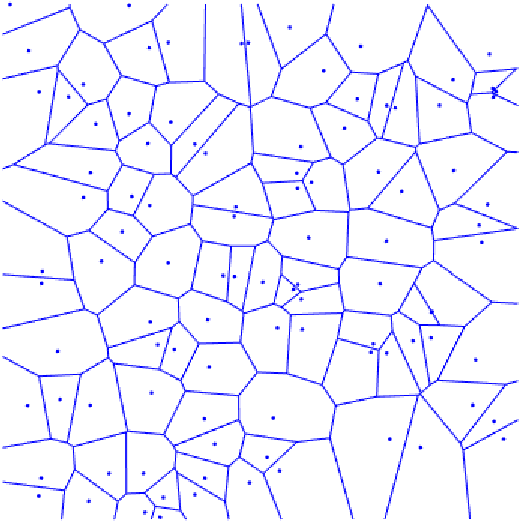
Random geometric graph



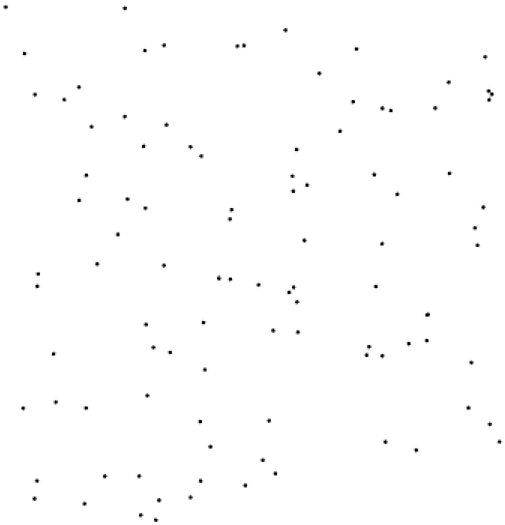
Make a deterministic geometric construction



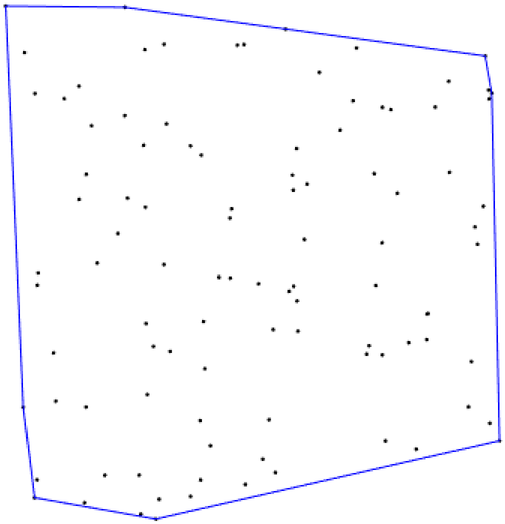
Poisson-Voronoi tessellation



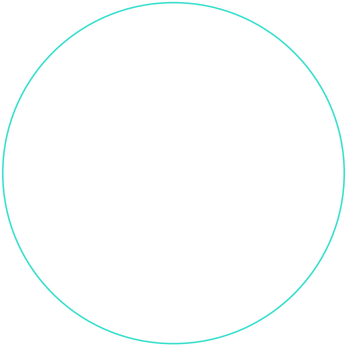
Make a deterministic geometric construction



Convex hull



Make an exact calculation



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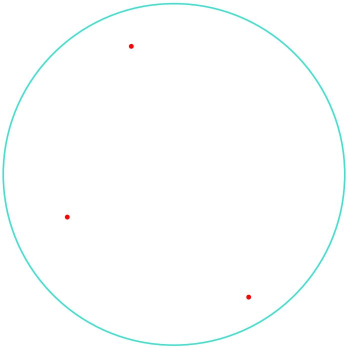
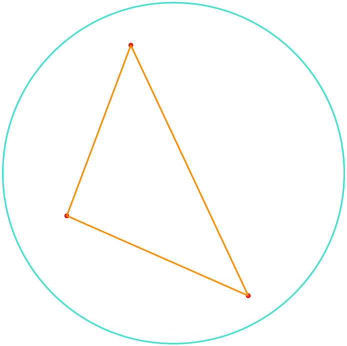
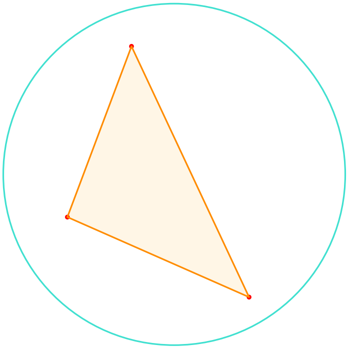


Make an exact calculation



Make an exact calculation



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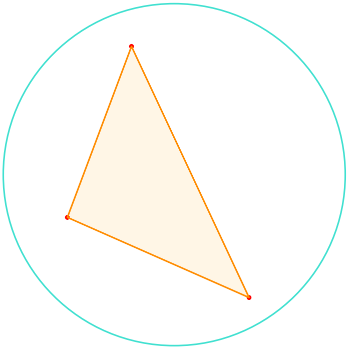
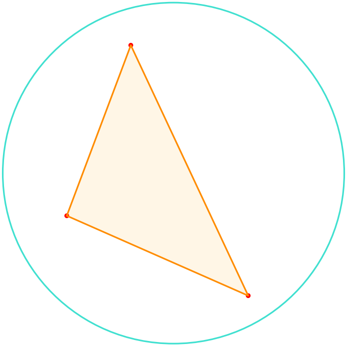


Mean area of the random triangle in the unit disk?

Make an exact calculation

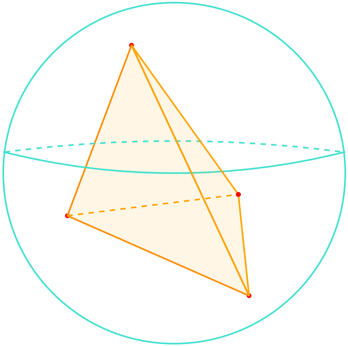


Mean area of the random triangle in the unit disk?

Answer: $\frac{35}{48\pi}$

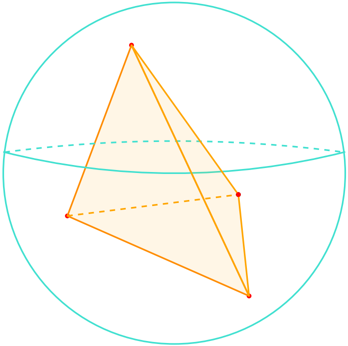
References. J. J. Sylvester (1864), W. S. B. Woolhouse (1867), W. Blaschke (1917)

Make an exact calculation



Mean volume of the random simplex in the unit ball in dimension d?

Make an exact calculation

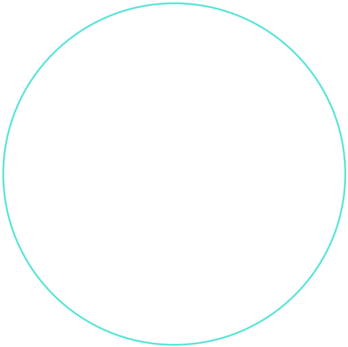


Mean volume of the random simplex in the unit ball in dimension d?

Answer: $$\frac{1}{\sqrt{\pi}\, d!}\left(\frac{d}{d+1}\right)^{d+1}\frac{\Gamma\left(\frac{d^2+2d+3}{2}\right)}{\Gamma\left(\frac{d^2+2d+2}{2}\right)}\frac{\Gamma\left(\frac{d}{2}\right)^{d+1}}{\Gamma\left(\frac{d+1}{2}\right)^{d}}$$

References. J. F. C. Kingman (1969), R. E. Miles (1971), Z. Kabluchko (2021)

Make an exact calculation



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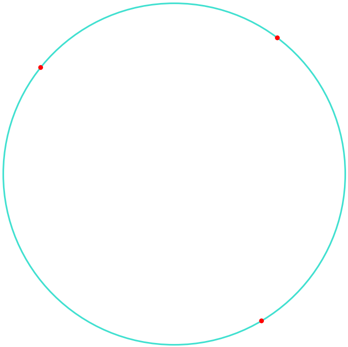

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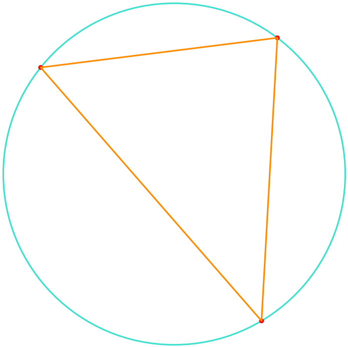

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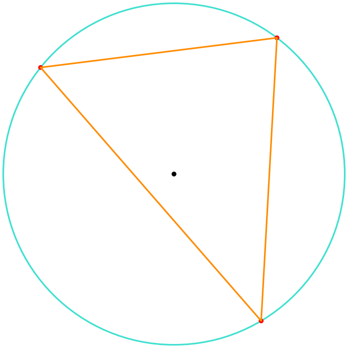

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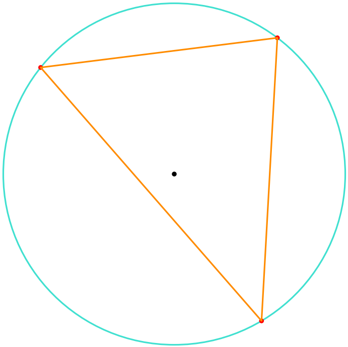
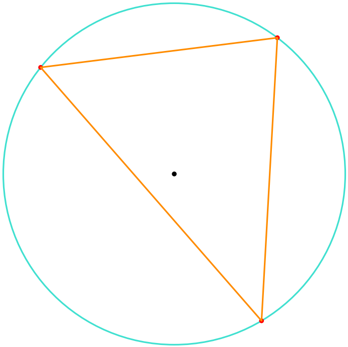


Probability that the center of the disk lies inside the triangle?

Make an exact calculation

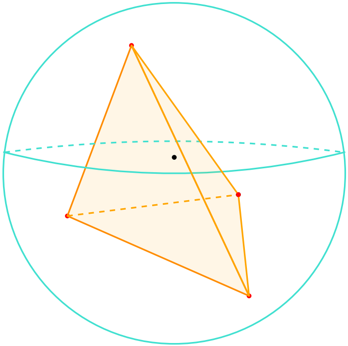


Probability that the center of the disk lies inside the triangle?

Answer: $\frac{1}{4}$

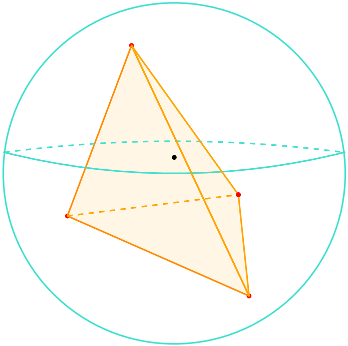
Reference. J. G. Wendel (1962)

Make an exact calculation



Probability that the center of the ball lies inside the convex hull of N points in dimension d?

Make an exact calculation

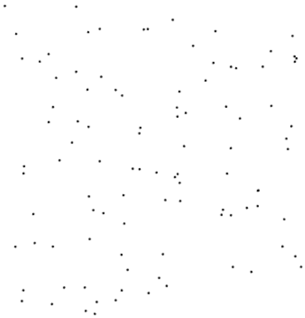


Probability that the center of the ball lies inside the convex hull of N points in dimension d?

Answer: $\mathbb{P}(S_{N-1} \geq d)$ where $S_{N-1} \stackrel{D}{=} Binomial(N-1, \frac{1}{2})$

Reference. J. G. Wendel (1962)

Make an exact calculation



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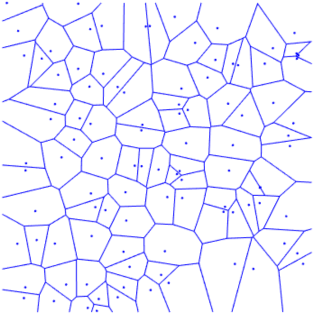

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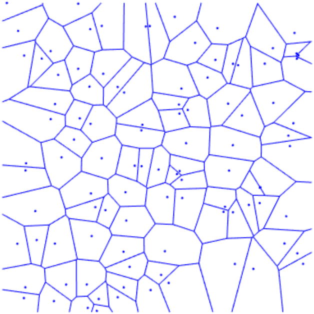
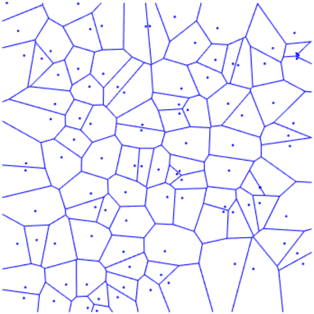


Mean number of vertices of a Voronoi cell picked at random?

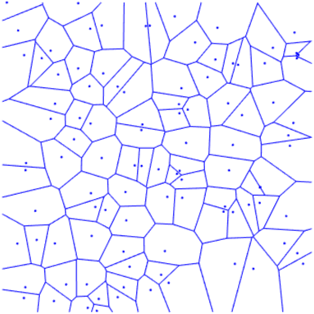
Make an exact calculation



Mean number of vertices of a Voronoi cell picked at random?

Answer: 6

Make an exact calculation

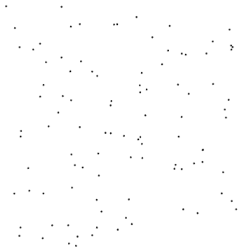


Mean number of vertices of a Voronoi cell picked at random in dimension d?

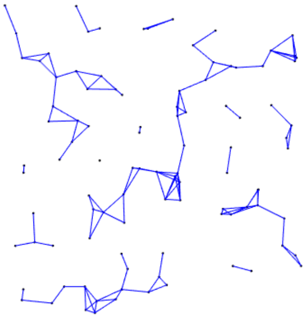
Answer: $2\pi^{\frac{d-1}{2}} d^{d-2} \left(\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\right)^d \frac{\Gamma(\frac{d^2+1}{2})}{\Gamma(\frac{d^2}{2})}$

Reference. J. Møller (1989)

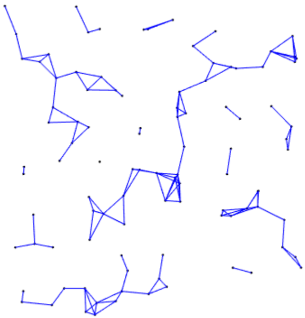
Study asymptotic problems



Study asymptotic problems



Study asymptotic problems



- Limit theorems for the total length of a graph

  References. F. Avram & D. Bertsimas (1993), K. S. Alexander (1996), J. Yukich (2012)

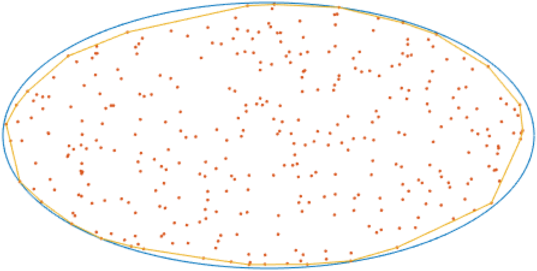
- Percolation, existence of infinite paths

  References. R. Meester & R. Roy (1996), J.-B. Gouéré (2008), F. Baccelli, D. Coupier & V. C. Tran (2016)

- Distribution of the degrees, maximal degree

  Reference. M. Penrose (1996)

Study asymptotic problems



- Geometric and combinatorial characteristics of large cells

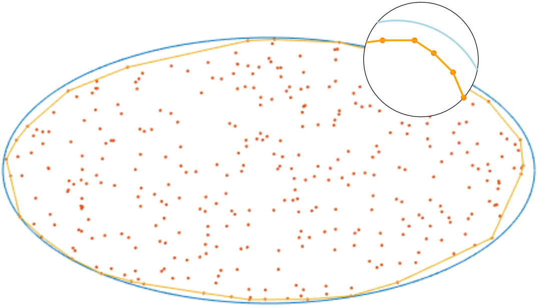
  References. A. Rényi & R. Sulanke (1963), I. Bárány (1989), M. Reitzner (2003)

- High dimension

  References. E. O'Reilly (2020), G. Bonnet, Z. Kabluchko & N. Turchi (2021), j.w. with B. Dadoun (2024)

- Close-up: limit shape of random convex sets and fluctuations

Study asymptotic problems



- Geometric and combinatorial characteristics of large cells

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- Close-up: limit shape of random convex sets and fluctuations

Plan

Some random geometry

Fluctuations of random convex hulls

Convex hull peeling

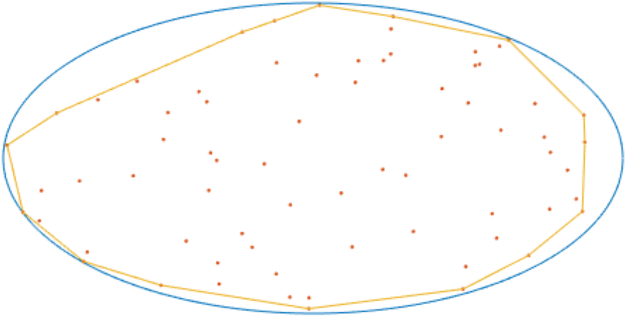
Joint works with **Joseph Yukich** and **Gauthier Quilan**

Random convex hull

K smooth convex body of $\mathbb{R}^d$

$\mathcal{P}_\lambda$ homogeneous Poisson point process of intensity λ in $\mathbb{R}^d$

K_λ convex hull of $\mathcal{P}_\lambda \cap K$



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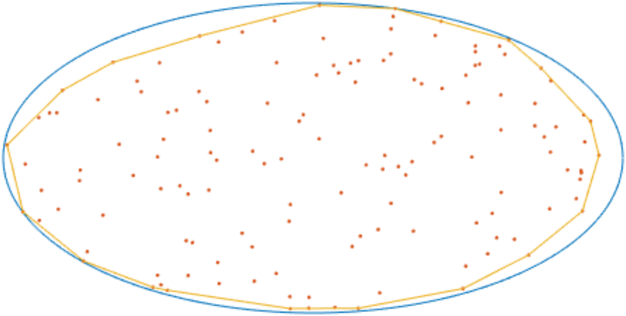

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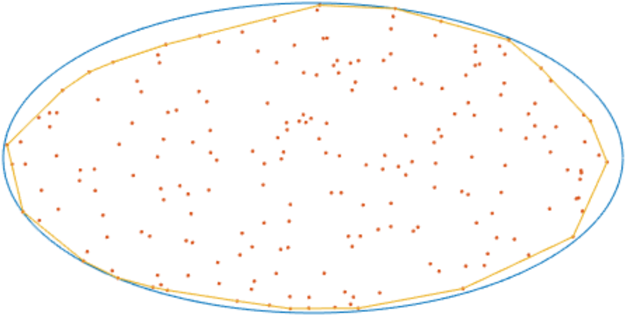

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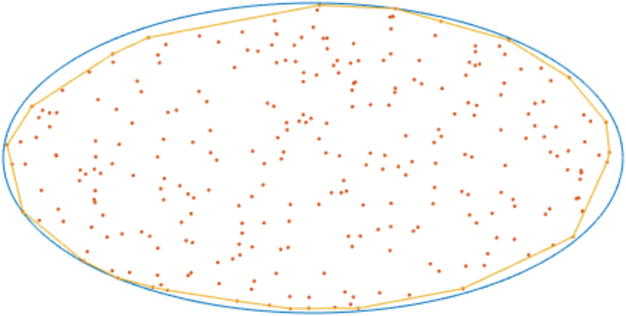

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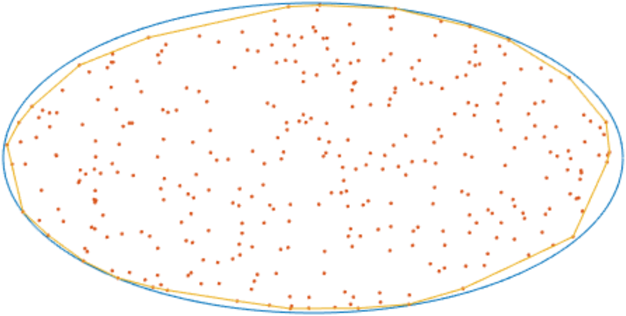

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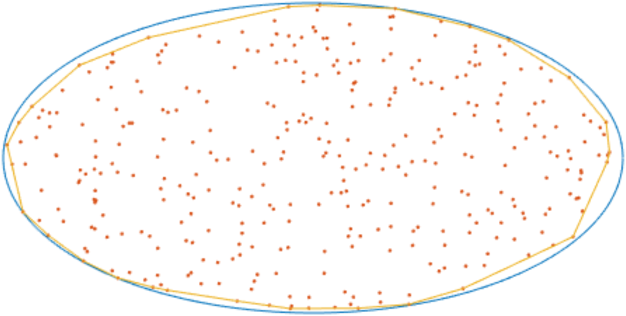
Fluctuations when $\lambda \to \infty$?



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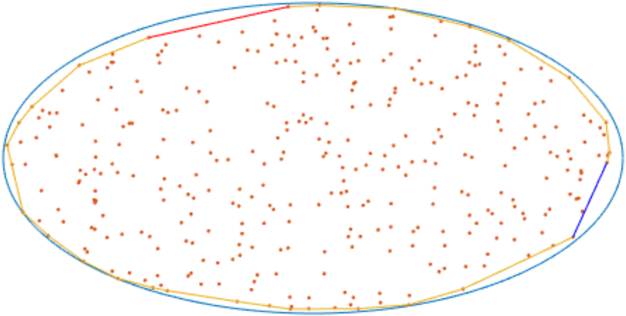

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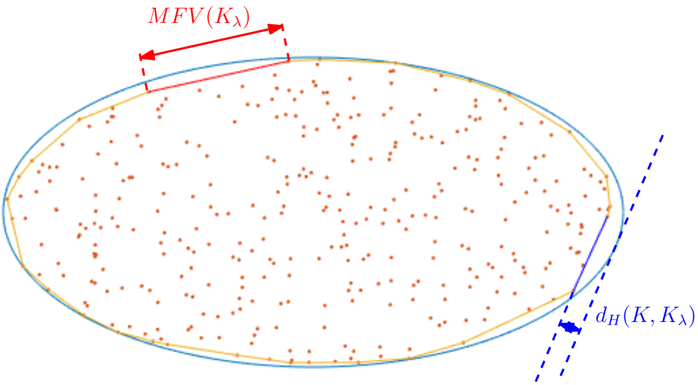
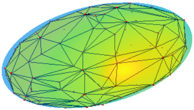


Facets of the random convex hull

Facets of K_n Simplices a.s.

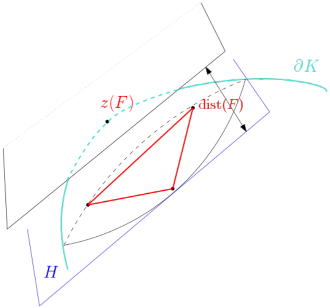


Mean number of facets $Z_\lambda \sim c(K)\lambda^{\frac{d-1}{d+1}}$ when $\lambda \to \infty$

Reference. H. Raynaud (1970)

Height and z-value of a facet

Each facet is included in a section of K by a hyperplane H.



$z(F) :=$ support point of the closest parallel hyperplane tangent to ∂K

dist$(F) :=$ distance between the two hyperplane

Radial fluctuation: Hausdorff distance

$$d_H(K, K_\lambda) := \min\{\varepsilon > 0 : K \subset K_\lambda + \varepsilon \mathbb{B}^d\},\ \mathbb{B}^d := \text{unit ball of } \mathbb{R}^d$$

References. I. Bárány (1989), H. Bräker, T. Hsing & N. H. Bingham (1998)

(with J. Yukich)

✓ $\left(a_o a_1 \frac{\log \lambda}{\lambda}\right)^{\frac{2}{d+1}} d_H(K, K_\lambda) \xrightarrow{\mathbb{P}} 1$

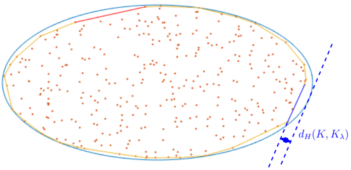
✓ $d_H(K, K_\lambda) = \lambda^{-\frac{2}{d+1}} (a_0(a_1 \log \lambda + a_2 \log(\log \lambda) + a_3 + \xi_\lambda))^{\frac{2}{d+1}}$

where $\mathbb{P}(\xi_\lambda \le t) \underset{\lambda \to \infty}{\longrightarrow} e^{-e^{-t}}$ (Gumbel distribution)

$$a_0 := \frac{\Gamma(\frac{d+3}{2}) \max_{\partial K} \kappa^{\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}}$$

$$a_1 := \frac{d-1}{d+1}$$

$\kappa :=$ Gauss curvature



Longitudinal fluctuation: maximal facet volume

$$MFV(K_\lambda) := \max_{F \text{ facet of } K_\lambda} \mathrm{Vol}_{d-1}(F),$$ $\mathrm{Vol}_{d-1} := (d-1)$-dimensional volume

(with J. Yukich)

✓ $\left(a_o a_1 \frac{\log \lambda}{\lambda}\right)^{\frac{d-1}{d+1}} MFV(K_\lambda) \xrightarrow{\mathbb{P}} 1$

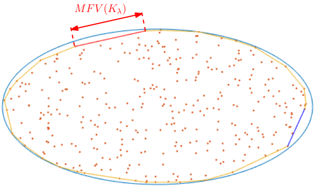
✓ $MFV(K_\lambda) = \lambda^{-\frac{d-1}{d+1}}(a_0(a_1 \log \lambda + a_2 \log(\log \lambda) + a_3 + \xi_\lambda))^{\frac{d-1}{d+1}}$

where $\mathbb{P}(\xi_\lambda \le t) \underset{\lambda\to\infty}{\longrightarrow} e^{-e^{-t}}$ (Gumbel distribution)

$$a_0 := \frac{2\Gamma(\frac{d+3}{2}) v_{d-1}^{\frac{d+1}{d-1}}}{\pi^{\frac{d-1}{2}} \min_{\partial K} \kappa^{\frac{1}{d-1}}}$$

$$a_1 := \frac{d-1}{d+1}$$

$$v_{d-1} := \mathrm{Vol}_{d-1}(\text{regular simplex in } \mathbb{B}^{d-1})$$



Strategy for an extreme value convergence

f	dist	Vol_{d-1}
α	$\frac{2}{d+1}$	$\frac{d-1}{d+1}$

$$f_\lambda(F) := a_0^{-1}\lambda f(F)^{\frac{1}{\alpha}} - (a_1 \log\lambda + a_2 \log\log\lambda + a_3),\ F \text{ facet of } K_\lambda$$

$\mathcal{F}_\lambda :=$ facet chosen at random, $Z_\lambda :=$ mean number of facets

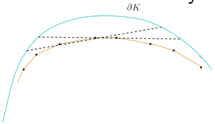
Aim $\max_{F\in\{\text{facets of } K_n\}} f_n(F) \xrightarrow{D}$??

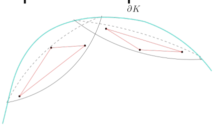
Prerequisite Convergence of $Z_\lambda \mathbb{P}(f_\lambda(\mathcal{F}_\lambda) \geq \tau)$ to $e^{-\tau}$

Use of the Poisson property and tools from integral geometry

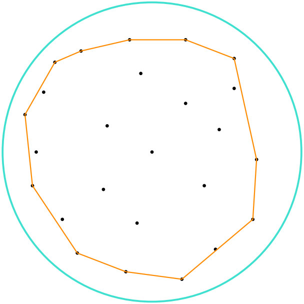
Mixing conditions

dist(F)	$\mathrm{Vol}_{d-1}(F)$
Exceedances by blocks	Asymptotic independence





Hausdorff distance for $K = \mathbb{B}^d$



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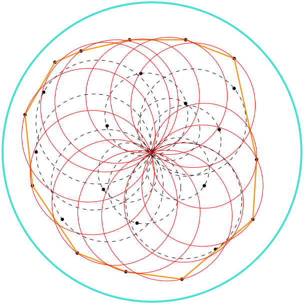

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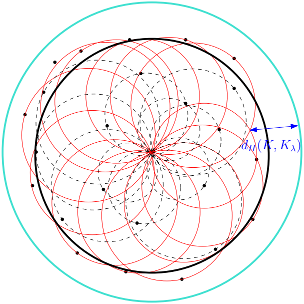

$(d_H(K, K_\lambda) \leq t_\lambda)$ iff $\partial(1 - t_\lambda)\mathbb{B}^d$ is covered by spherical caps $B(\frac{x}{2}, \frac{\|x\|}{2}) \cap \partial(1 - t_\lambda)\mathbb{B}^d$, $x \in \mathcal{P}_\lambda$.

Poisson number Λ of caps with a radius $\propto \varepsilon$. If

$$c_1 \varepsilon^{d-1} \Lambda + (d-1)\log(\varepsilon) - (d-1)\log(-\log(\varepsilon)) + c_2 \underset{\varepsilon \to 0}{\longrightarrow} u,$$

then the covering probability converges to $\exp(-e^{-u})$.

Reference. S. Janson (1986)

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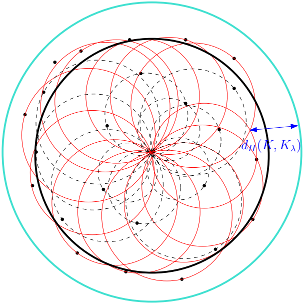

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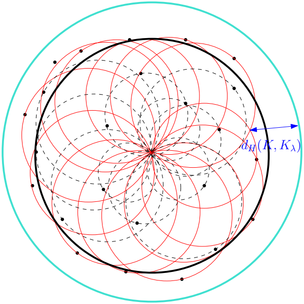

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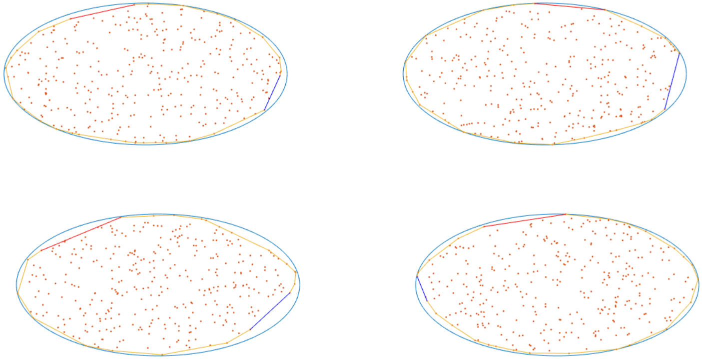
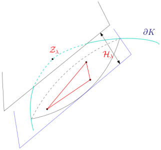
Location of the maxima

max $\mathrm{dist}(F)$ reached near max κ, max $\mathsf{Vol}_{d-1}(F)$ near min κ



Location of the maximal facet volume

$F_{\lambda,\max} := \operatorname{argmax}(\mathrm{Vol}_{d-1}(\cdot))$



$\mathcal{Z}_\lambda :=$ associated support point

(with J. Yukich) $\kappa(\mathcal{Z}_\lambda) \xrightarrow[\lambda\to\infty]{D} \min_{z\in\partial K} \kappa(z)$

If $\mathrm{Vol}_{d-1}(\operatorname{argmin}(\kappa)) > 0$

$$\mathcal{Z}_\lambda \xrightarrow[\lambda\to\infty]{D} \text{Unif}\,(\operatorname{argmin}(\kappa)).$$

If $\operatorname{argmin}(\kappa) = \{z_1, \cdots, z_k\}$,

$$\mathcal{Z}_\lambda \xrightarrow[\lambda\to\infty]{D} \sum_{i=1}^{k} w_i \delta_{z_i}$$

where $w_i \propto \left(\det(D^2\kappa\big|_{z_i})\right)^{-\frac{1}{2}}$.

- Extension when $0 < \dim(\operatorname{argmin}(\kappa)) < d-1$
- Limit shape of $F_{\lambda,\max}$: regular simplex up to rescaling
- Analogous results for $f = \text{dist}$

Tracy-Widom like distribution

Definition $F_{TW}(t) := \exp\left(-\int_t^\infty (x-t)q(x)^2 \mathrm{d}x\right)$
where q is the solution of the Painlevé II ODE $q'' = xq + 2q^3$ with asymptotics given by the Airy function.

GUE eigenvalues

$$n^{\frac{1}{6}}(\lambda_n - 2\sqrt{n}) \xrightarrow{D} F_{TW}$$

where $\lambda_n :=$ largest eigenvalue of a GUE random matrix

References. C. Tracy & H. Widom (1994)

Tails

$$1 - F_{TW}(t) \underset{t\to\infty}{\sim} t^{-\frac{3}{2}} e^{-\frac{4}{3}t^{\frac{3}{2}}} \quad \Big| \quad \mathbb{P}(\lambda^{-\frac{1}{3}}(\lambda \mathrm{dist}(\mathcal{F}_\lambda)) \geq t) \sim c t^{\frac{3}{2}} e^{-\frac{4\sqrt{2}}{3\pi}t^{\frac{3}{2}}}$$

The typical height fluctuations exhibit Tracy-Widom like tails.

Comparison with the KPZ universality class

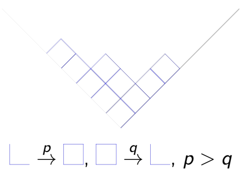
References. I. Corwin (2012), K. Matetski, J. Quastel & D. Remenik (2021)

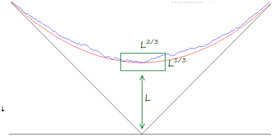
KPZ equation $\partial_t h = \lambda(\partial_x h)^2 + \nu\partial_x^2 h + \sigma\xi$, $\xi :=$ space-time white noise

Class of growth models involving a random height function $h(x, t)$ with

- linear growth
- $t^{\frac{1}{3}}$ fluctuations with GUE Tracy-Widom limit
- $t^{\frac{2}{3}}$ spatial correlation

Examples. Partially asymmetric corner growth model, TASEP, directed polymers





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When $K = \mathbb{B}^2$, $(\lambda\partial K_\lambda)$ shares common features with the KPZ class.

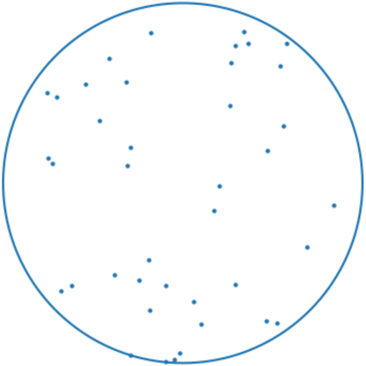
Plan

Some random geometry

Fluctuations of random convex hulls

Convex hull peeling

Iterated construction: *Convex hull peeling*



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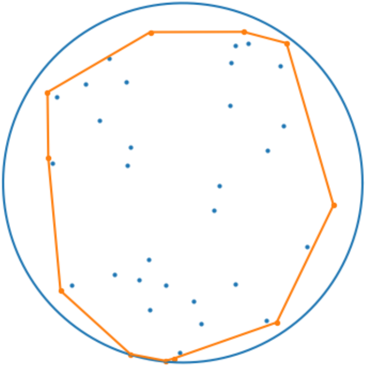

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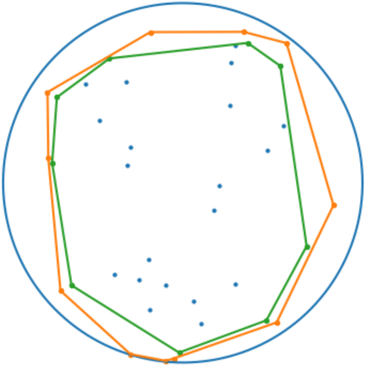

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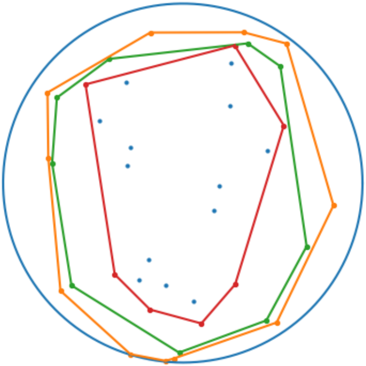

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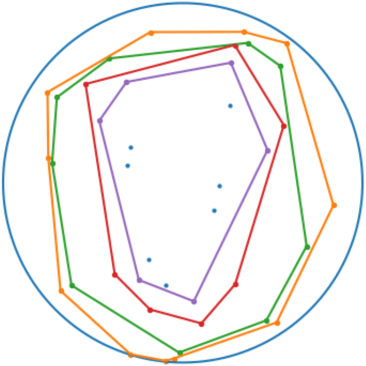

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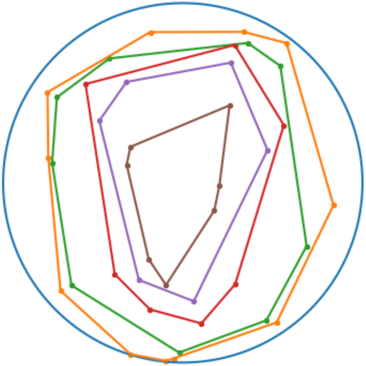


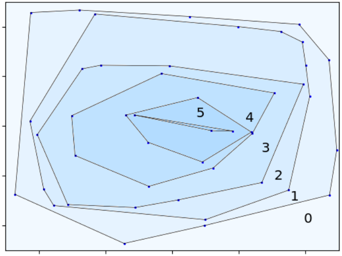
Convex height function

K convex body of $\mathbb{R}^d$

$\mathcal{P}_\lambda$ homogeneous Poisson point process of intensity λ in $\mathbb{R}^d$

Layer of label n $\mathrm{Conv}_n(\mathcal{P}_\lambda \cap K) :=$ convex hull at step n of the peeling

Height function $h_\lambda := \sum_{n \geq 1} \mathbf{1}(\mathrm{int}(\mathrm{Conv}_n(\mathcal{P}_\lambda \cap K)))$



Asymptotic estimate of the height function

K. Dalal (2004)

✓ Monotonicity of the height function with respect to the point set

✓ $\mathbb{E}(\max h_\lambda) = \Theta(\lambda^{\frac{2}{d+1}})$ for every K

J. Calder & **C. K. Smart** (2020)

✓ Uniform convergence in probability of $\lambda^{-\frac{2}{d+1}} h_\lambda$ to ch (c only depends on d)

✓ The function h is the unique viscosity solution of

$$\begin{cases} \langle Dh,\ {}^t\mathrm{com}(-D^2 h) Dh \rangle = f^2 \text{ in } \mathrm{int}(K) \\ h = 0 \text{ on } \partial K \end{cases},$$

where f is the common density of the points from the input

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Dual approach: number of points on each layer

$K = \mathbb{B}^d$

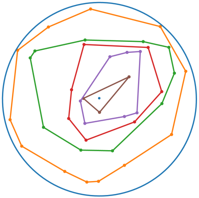
$f_k(\mathrm{Conv}_n(\mathcal{P}_\lambda \cap \mathbb{B}^d))$:=number of k-dimensional faces of the n-th layer

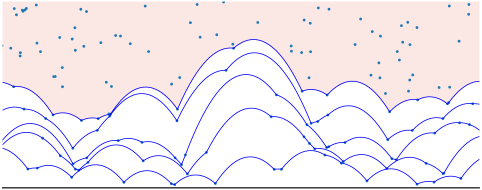
(with G. Quilan)

✓ $\mathbb{E}(f_k(\mathrm{Conv}_n(\mathcal{P}_\lambda \cap K))) \underset{\lambda\to\infty}{\sim} c(d,n,k)\lambda^{\frac{d-1}{d+1}}$ for every $n \geq 1$

✓ Limiting variances and Gaussian limit distributions

✓ Same results for the volume of the difference $\mathbb{B}^d \setminus \mathrm{Conv}_n(\mathcal{P}_\lambda \cap K)$





Dual approach: number of points on each layer

K = simple polytope

$f_k(\mathrm{Conv}_n(\mathcal{P}_\lambda \cap K))$:=number of k-dimensional faces of the n-th layer

(with G. Quilan)

✓ $\mathbb{E}(f_k(\mathrm{Conv}_n(\mathcal{P}_\lambda \cap K))) \underset{\lambda\to\infty}{\sim} c(d,n,k)\log(\lambda)^{d-1}$ for every $n \geq 1$

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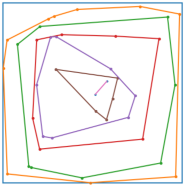

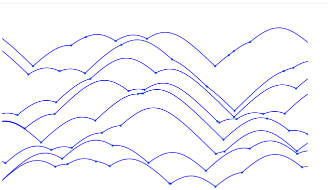

Thank you for your attention!