Sur les Graphes Aléatoires Unimodulaires

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Outline

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Unimodular Random Graphs

- A **graph** G with set of vertices $V(G)$
- A **rooted** graph: $[G, o] \in \mathcal{G}_*$
  - o: the **origin** or the **root**
  - each node has finite degree (locally finite)
- A **random** graph: $[\boldsymbol{G}, \boldsymbol{o}]$
- **Unimodular** if (heuristically) "*$\boldsymbol{o}$ is uniformly distributed in $\boldsymbol{G}$*"

$$\forall g : \mathbb{E}\left[\sum_{v \in V(\boldsymbol{G})} g[\boldsymbol{G}, \boldsymbol{o}, v]\right] = \mathbb{E}\left[\sum_{v \in V(\boldsymbol{G})} g[\boldsymbol{G}, v, \boldsymbol{o}]\right] \quad \text{(mtp)}$$

Extends to boundedly finite **random rooted discrete metric spaces**.

Outline

Example 1: Palm probabilities

Palm version of stationary point processes

- A random discrete subsets of $\mathbb{R}^k$
- Distribution invariant under translations
- Conditioned on containing the origin

All covariant graphs on stationary point processes under their Palm version are unimodular

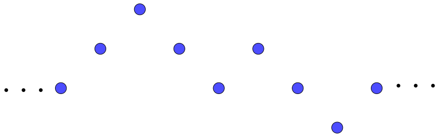
All stationary point processes under their Palm version are unimodular random discrete metric spaces

Example 2: Graph of a process with stationary increments

$\{X_n\}_{n\in\mathbb{Z}}$ a **stationary stochastic process** with values on $\mathbb{R}^d$

$$
\begin{aligned}
S_0 &= 0, \quad S_i - S_{i-1} = X_{i-1},\ i \in \mathbb{Z} \\
S_i &= \sum_{n=0}^{i-1} X_n,\ i > 0, \quad S_i = -\sum_{n=-i}^{-1} X_n,\ i < 0
\end{aligned}
$$

The **graph** $[\mathbf{G}, (\mathbf{0}, \mathbf{0})]$ with $\mathbf{G} = \{i, S_i\}_i$ **is unimodular**



Example 3: Finite graph limits

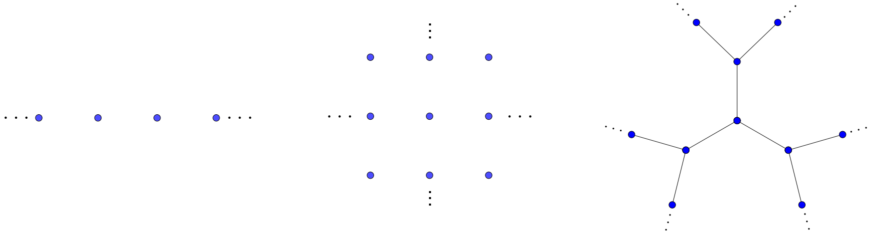
An **a.s. finite random graph with a root picked at random** in the set of vertices is a unimodular rooted discrete space for graph distance

A **local weak limit** of such a random rooted graph is unimodular **[Aldous Lyons 07]**

Canopy Tree Example

- Binary tree with say N generations
- Choose a root o_N at random and let N tend to infinity
- The local weak limit is the **Canopy Tree** which has infinitely many generations, numbered like $\mathbb{N}$
- The index (w.r.t. the generation of the root) of the last generation in this limit is geometrically distributed with parameter $1/2$

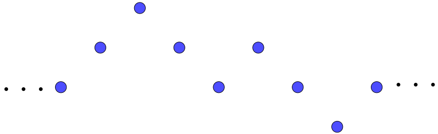
Examples 4: Deterministic graphs



Lattices of $\mathbb{R}^k$
Cayley graphs of finitely generated groups

Example 6: Point-Stationary Point Processes

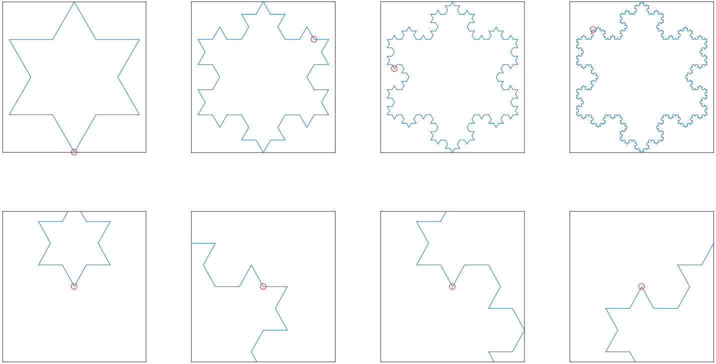
- The zeros or the graph of the simple random walk.



- Point-stationary point processes.

Example 7: "Fractals"

- Unimodular discrete Koch snowflake.



- Local weak convergence to a unimodular discrete limit
- Extends to a wide class of self similar unimodular discrete spaces

Outline

1. Unimodular Random Graphs
   - Examples
   - Covariant Processes
2. Unimodular Minkowski and Hausdorff Dimensions
3. Applications: Classification of Dynamics on Unimodular Random Graphs
4. EFTs Everywhere
5. Conclusions

Covariant process

G: a deterministic locally-finite graph

Marking of G: with values in Ξ a measure space

- a function from $E(G)$ to Ξ (edge marking)
- a function from $V(G)$ to Ξ (vertex marking)

Covariant process $\boldsymbol{Z}$ with values in Ξ
map that assigns to every G a random marking $\boldsymbol{Z}_G$ s.t.

(i) $\boldsymbol{Z}$ is compatible with isomorphisms: $\forall$ isomorphisms $\rho : G_1 \to G_2$, $\boldsymbol{Z}_{G_1} \circ \rho^{-1}$ of G_2 has the same distribution as $\boldsymbol{Z}_{G_2}$

(ii) For every measurable subset $A \subseteq \mathcal{G}'_*$, the function

$$[G, o] \mapsto \mathbb{P}\left[[G, o; \boldsymbol{Z}_G] \in A\right]$$

is measurable

Marked Graphs

Lemma
Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular discrete space.
If $\boldsymbol{Z}$ is a covariant process on $\boldsymbol{G}$, then $[\boldsymbol{G}, \boldsymbol{o}; \boldsymbol{Z}_{\boldsymbol{G}}]$ is also unimodular

Examples:

- **Deterministic**: in a one ended tree, mark each edge incident to a node with its direction to the end
- **Random**: in a graph, declare the directed edge from a node to one of its neighbors independently for all neighbors but with a probability that depends on the degree of the node

Marked unimodular graphs are called **networks [Aldous, Lyons 07]**

Covariant subset, Intensity, Covariant subgraph

Definition: Covariant subset:
Set $\boldsymbol{S} = \mathbf{S_G}$ of nodes with mark 1 in some $\{0,1\}$-valued covariant process of $[\boldsymbol{G}, \boldsymbol{o}]$.

Definition: Intensity:
If $[\boldsymbol{G}, \boldsymbol{o}]$ is a unimodular random graph, then the **intensity** of $\boldsymbol{S}$ in $\boldsymbol{G}$ is

$$\rho_{\boldsymbol{G}}(\boldsymbol{S}) := \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{G}}]$$

Definition: Covariant subgraph:

- $\boldsymbol{H}_{\boldsymbol{G}}$: the restriction of $\boldsymbol{G}$ to $\boldsymbol{S}_{\boldsymbol{G}}$
- $\mathbb{P}_{\boldsymbol{H}}$: $\mathbb{P}$ conditioned on $\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{G}}$

Under $\mathbb{P}_{\boldsymbol{H}}$, $[\boldsymbol{H}_{\boldsymbol{G}}, \boldsymbol{o}]$ is a unimodular random graph.

Unimodularity as a Continuation of Palm Calculs

- Intensity extends the same notion for stationary point processes
- Covariant subsets/subgraphs extend the notion of thinning of stationary point processes

Many other notions "extend" Palm calculus of the Euclidean space to unimodular discrete spaces

- Ergodic theory like results
- Exchange formulas

Poincaré Recurrence Lemma

Unimodular Poincaré recurrence lemma, **[BHK18], [Lovász 20]**

Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular network s.t. $V(\boldsymbol{G})$ is a.s. infinite.
Then any covariant subset $\boldsymbol{S}$ of $V(\boldsymbol{G})$ is a.s. **either empty or infinite**:

$$\mathbb{P}\left[\#\boldsymbol{S}_{\boldsymbol{G}} \in \{0, \infty\}\right] = 1$$

Unimodular extension of a classical result on stationary point processes

Several other **unimodular extensions of the theory of measure preserving transformations** hold

Proof of unimodular Poincaré recurrence lemma

Preliminary lemma

Let $[\boldsymbol{G},\boldsymbol{o}]$ be a unimodular network and $\boldsymbol{S}=\boldsymbol{S_G}$ be a covariant subset of $V(\boldsymbol{G})$. Then $\mathbb{P}[\boldsymbol{S_G}\neq\emptyset]>0$ iff $\mathbb{P}[o\in\boldsymbol{S_G}]>0$.

Proof

Let $g(G,o,s)=1_{s\in S_G}$. Assume $\mathbb{P}[S_{\mathbf{G}}\neq\emptyset]>0$. By MTP,

$$\begin{aligned}0<\mathbb{E}\left[\#S_{\boldsymbol{G}}\right] &= \mathbb{E}\left[\sum_{s\in V(\boldsymbol{G})} g[\boldsymbol{G},\boldsymbol{o},s]\right]\\ &= \mathbb{E}\left[\sum_{s\in V(\boldsymbol{G})} g[\boldsymbol{G},s,\boldsymbol{o}]\right]=\mathbb{E}\left[1_{\{\boldsymbol{o}\in S_{\boldsymbol{G}}\}}\#V(\boldsymbol{G})\right].\end{aligned}$$

Therefore, $\mathbb{P}\left[\boldsymbol{o}\in S_{\boldsymbol{G}}\right]>0$. The converse is clear.

Proof of unimodular Poincaré recurrence lemma

Assume that, with a positive probability, $0 < \#\boldsymbol{S_G} < \infty$. Let

$$g(\boldsymbol{G}, \boldsymbol{o}, v) = \frac{1}{\#\boldsymbol{S_G}} 1_{v \in \boldsymbol{S_G}} 1_{0 < \#\boldsymbol{S_G} < \infty}$$

We have $g^+(o) \leq 1$ and $g^-(o) = \frac{\#\boldsymbol{G}}{\#\boldsymbol{S_G}} 1_{o \in \boldsymbol{S_G}} 1_{0 < \#\boldsymbol{S_G} < \infty}$
But $\mathbb{P}[\{0 < \#\boldsymbol{S_G} < \infty\} \cap \{0 \in \boldsymbol{S_G}\}] = \mathbb{P}[o \in \widetilde{\boldsymbol{S}}_{\boldsymbol{G}}]$ with

$$\widetilde{\boldsymbol{S}}_{\boldsymbol{G}} = \boldsymbol{S_G} 1_{0 < \#\boldsymbol{S_G} < \infty} + \emptyset 1_{\#\boldsymbol{S_G} = \infty}, \quad \text{covariant subset}$$

By the preliminary lemma, $\mathbb{P}[o \in \widetilde{\boldsymbol{S}}_{\boldsymbol{G}}] > 0$. It follows that $g^+(0) = \infty$ with positive proba. $\square$

Exchange formula

Proposition [BHK18]
Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular network and $\boldsymbol{H}$ and $\boldsymbol{H}'$ be two covariant subnetworks associated with the covariant subsets $\boldsymbol{S}$ and $\boldsymbol{S}'$. For all measurable functions $g : \mathcal{G}_{**} \to \mathbb{R}^{\geq 0}$,

$$\rho_{\boldsymbol{G}}(\boldsymbol{S})\mathbb{E}_{\boldsymbol{H}}\left[\sum_{v' \in V(\boldsymbol{H}'_{\boldsymbol{G}})} g[\boldsymbol{G}, \boldsymbol{o}, v']\right] = \rho_{\boldsymbol{G}}(\boldsymbol{S}')\mathbb{E}_{\boldsymbol{H}'}\left[\sum_{v \in V(\boldsymbol{H}_{\boldsymbol{G}})} g[\boldsymbol{G}, v, \boldsymbol{o}]\right].$$

Proof
Let $\hat{g}(\boldsymbol{G}, v, w) := 1_{\{v \in V(\boldsymbol{H}_{\boldsymbol{G}})\}} 1_{\{w \in V(\boldsymbol{H}'_{\boldsymbol{G}})\}} g[\boldsymbol{G}, v, w]$. The claim is a direct implication of MTP for $\hat{g}$. $\square$

Unimodular extension of the exchange formula between two Palm probabilities

Outline

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- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular random discrete metric space (for instance a graph).
- A **covariant** r-**covering** (not depending on $\boldsymbol{o}$):
  A **covariant subset** $\boldsymbol{S} \subseteq \boldsymbol{D}$ s.th.

$$\bigcup_{v \in \boldsymbol{S}} N_r(v) = \boldsymbol{D}$$

- Heuristically, $\#\boldsymbol{S}$ in $\boldsymbol{D} \sim \mathbb{E}\left[1_{\{\boldsymbol{o} \in \boldsymbol{S}\}}\right] =: \rho_{\boldsymbol{D}}(\boldsymbol{S})$ the **intensity** of $\boldsymbol{S}$.

Definition

$$\lambda_r := \inf\{\text{intensity of } \boldsymbol{S} : \boldsymbol{S} \text{ is a covariant } r\text{-covering of } \boldsymbol{D}\}$$

$$\overline{\mathrm{udim}}_M(\boldsymbol{D}) := \overline{\mathrm{decay}}\,(\lambda_r) := \limsup_{r \to \infty} \frac{-\log \lambda_r}{\log r}$$

$$\underline{\mathrm{udim}}_M(\boldsymbol{D}) := \underline{\mathrm{decay}}\,(\lambda_r) := \liminf_{r \to \infty} \frac{-\log \lambda_r}{\log r}$$

- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular discrete space

Theorem [BHK21]
There exists an optimal r-covering; i.e, the infimum in λ_r is attained.

Theorem [BHK21]
A **disjoint** r-covering is optimal.

- Proof:
  - $\boldsymbol{S}$: a disjoint r-covering, $\boldsymbol{S}'$: arbitrary r-covering.

$$g(u,v) := 1_{\{u \in \boldsymbol{S}\}} 1_{\{v \in \boldsymbol{S}'\}} 1_{\{d(u,v) \leq r\}}$$

$$\rho(\boldsymbol{S}') = \mathbb{E}\left[1_{\{\boldsymbol{o} \in \boldsymbol{S}'\}}\right] = \mathbb{E}\left[\sum_u g(u, \boldsymbol{o})\right] = \mathbb{E}\left[\sum_v g(\boldsymbol{o}, v)\right] \geq \mathbb{E}\left[1_{\{\boldsymbol{o} \in \boldsymbol{S}\}}\right] = \rho(\boldsymbol{S})$$

- $\mathrm{udim}_M(\mathbb{Z}^k) = k$.

Outline

Unimodular Hausdorff Dimension

- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular discrete space.
- **Covariant covering**:
  - The radius of the ball centered at $v =: \boldsymbol{R}(v)$.
  - Extra randomness is allowed.
  - $\boldsymbol{R}(v) = 0$ means no ball.
  - The balls cover $\boldsymbol{D}$.
- Heuristically, $\sum_v \boldsymbol{R}(v)^\alpha \sim \mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right]$.

Definition

$$\begin{aligned} \mathcal{H}_M^\alpha(\boldsymbol{D}) &:= \inf\{\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] : \boldsymbol{R} \text{ is a covariant covering} \\ &\qquad \text{and } \boldsymbol{R}(\cdot) \in \{0\} \cup [M, \infty) \ a.s.\} \\ \mathrm{udim}_H(\boldsymbol{D}) &:= \sup\{\alpha \geq 0 : \mathcal{H}_1^\alpha(\boldsymbol{D}) = 0\} \end{aligned}$$

First Results

Theorem [BHK21] If $[\boldsymbol{D}, \boldsymbol{o}]$ is a unimodular discrete space, then

$$\underline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \overline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \mathrm{udim}_H(\boldsymbol{D})$$

Proof of rightmost inequality
By the definition of λ_r in Minkowski, $\forall\ \alpha \geq 0$ and $r \geq 1$,

$$\inf\{\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] : \boldsymbol{R} \text{ is an equivariant } r\text{-covering}\} = r^\alpha \lambda_r.$$

Hence $\mathcal{H}_1^\alpha(\boldsymbol{D}) \leq r^\alpha \lambda_r$ for every $r \geq 1$. So, if $\alpha < \overline{\mathrm{decay}}\,(\lambda_r)$, $\mathcal{H}_1^\alpha(\boldsymbol{D}) = 0$.

Theorem [BHK21]
If $\boldsymbol{S}$ is a nonempty covariant subset of $\boldsymbol{D}$, then

$$\mathrm{udim}_H(\boldsymbol{S}) = \mathrm{udim}_H(\boldsymbol{D})$$

$$\overline{\mathrm{udim}}_M(\boldsymbol{S}) \geq \overline{\mathrm{udim}}_M(\boldsymbol{D}) \qquad \underline{\mathrm{udim}}_M(\boldsymbol{S}) \geq \underline{\mathrm{udim}}_M(\boldsymbol{D})$$

Unimodular Mass Distribution Principle

- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular discrete space.
- $\boldsymbol{w} : \boldsymbol{D} \to \mathbb{R}^{\geq 0}$: an **covariant weight function** not identical to zero.
- For $S \subseteq \boldsymbol{D}$, $\boldsymbol{w}(S) := \sum_{v \in S} \boldsymbol{w}(v)$.

Theorem [BHK21]
If $\forall r > 1 : \boldsymbol{w}(N_r(\boldsymbol{o})) \leq cr^{\alpha}$ a.s., then $\mathrm{udim}_H(\boldsymbol{D}) \leq \alpha$

Proof $\boldsymbol{R}$: a covariant covering

- $g(u, v) := w(v) 1_{\{\boldsymbol{R}(u) \neq 0\}} 1_{\{v \in N_{\boldsymbol{R}(u)}(u)\}}$

$$
\begin{aligned}
c\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^{\alpha}\right] &\geq \mathbb{E}\left[w(N_{\boldsymbol{R}(\boldsymbol{o})}(\boldsymbol{o}))\right] = \mathbb{E}\left[\sum_{v} g(\boldsymbol{o}, v)\right] \\
&= \mathbb{E}\left[\sum_{u} g(u, \boldsymbol{o})\right] \geq \mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right]
\end{aligned}
$$

- $\mathcal{H}_1^{\alpha}(\boldsymbol{D}) \geq \frac{1}{c}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right] > 0 \Rightarrow \mathrm{udim}_H(\boldsymbol{D}) \leq \alpha$.

Unimodular Billingsley Lemma

- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular discrete space
- $\boldsymbol{w} : \boldsymbol{D} \to \mathbb{R}^{\geq 0}$: a covariant weight function s.th. $0 < \mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] < \infty$
- $\overline{\mathrm{growth}}(\boldsymbol{w}) := \limsup \frac{\log \boldsymbol{w}(N_r(\boldsymbol{o}))}{\log r}$. $\underline{\mathrm{growth}}(\boldsymbol{w}) := \liminf$.

Theorem [BHK21]

If $\boldsymbol{w}$ has constant growth rates, then

$$\underline{\mathrm{growth}}(\boldsymbol{w}) \leq \mathrm{udim}_H(\boldsymbol{D}) \leq \overline{\mathrm{growth}}(\boldsymbol{w})$$

In general,

$$\text{ess inf } \underline{\mathrm{growth}}(\boldsymbol{w}) \leq \mathrm{udim}_H(\boldsymbol{D}) \leq \text{ess inf } \overline{\mathrm{growth}}(\boldsymbol{w})$$

Unimodular Frostman Lemma

- $[\boldsymbol{D}, \boldsymbol{o}]$: a unimodular discrete space

  Theorem [BHK21]
  If $\mathrm{udim}_H(\boldsymbol{D}) < \alpha$, there exists a covariant weight function $\boldsymbol{w}$ such that

$$\forall r \geq 1 : \boldsymbol{w}(N_r(\boldsymbol{o})) \leq r^{\alpha}$$

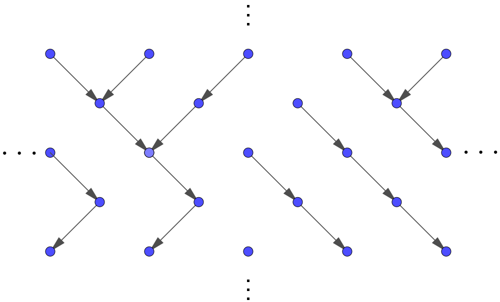
Outline

Examples

- If $\boldsymbol{D}$ is finite with positive probability, then $\mathrm{udim}_H(\boldsymbol{D}) = 0$.
- $\mathrm{udim}_H(\mathbb{Z}^k) = k$.
- A stationary point process in $\mathbb{R}^k \Rightarrow \mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D}) = k$.
- Zeros of SRW: $\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D}) = \frac{1}{2}$.
- Unimodular disc. Koch snowflake: $\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D}) = \frac{\log 4}{\log 3}$.
- A point-stationary point process in $\mathbb{R}^k \Rightarrow \mathrm{udim}_H(\boldsymbol{D}) \leq k$.
- Cayley graphs: $\mathrm{udim}_M(\boldsymbol{G}) = \mathrm{udim}_H(\boldsymbol{G}) =$ polynomial growth rate $\in \mathbb{Z} \cup \{\infty\}$ (Gromov's theorem).

Examples

- A river network model.



$$\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D}) = \frac{3}{2}.$$

Examples: General Unimodular Trees

Theorem: The number of **ends** of a unimodular tree is 0, 1, 2 or ∞ a.s.

- 0 end: $\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{T}) = 0$.
- 2 ends: $\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{T}) = 1$.
- ∞ many ends: $\mathrm{udim}_H(\boldsymbol{T}) = \infty$, $\mathrm{udim}_M(\boldsymbol{T})$ may be finite or infinite.
- 1 end:

$$
\begin{aligned}
\overline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) \geq n\right]\right) \\
\underline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \underline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) \geq n\right]\right) \\
\mathrm{udim}_H(\boldsymbol{T}) &\geq \overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) = n\right]\right)
\end{aligned}
$$

with $h(\boldsymbol{o})$ the height of the trees of descendants of the root.

Outline

Dynamics on a Discrete Random Structure

Dynamics: point-shifts, vertex-shifts

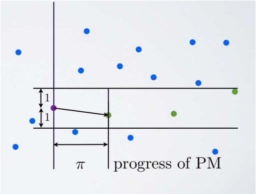
1. **Select one node as image of the root** (point/vertex)
   - **in** the discrete rooted structure
   - **as a covariant function of** the discrete rooted structure
2. **Move the origin/root there**

Point-shifts in the literature

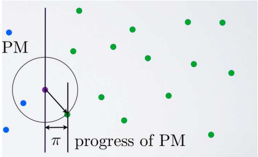
- **[Mecke 75]**: mass stationarity
- **[Thorisson 00]**: this terminology
- **[Holroyd & Peres 05]**: allocation rule

Examples of Point-Shifts on Poisson Point Processes

Strip Routing PS on $\mathbb{R}^2$
[Ferrari, Landim & Thorisson 05]



Directional PS on $\mathbb{R}^2$
[F.B. & Bordenave 07]
(radial spanning tree)



Locally defined "navigation rule" on the support of the point process

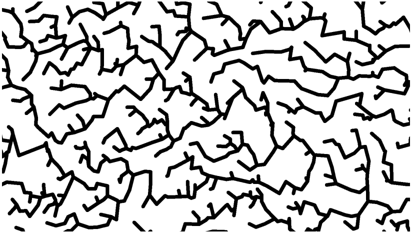
Mecke's Invariance Theorem

Theorem [J. Mecke 75]
Let f be a point-shift on a stationary point process Φ. Then θ_f preserves the Palm distribution of Φ if and only if f is almost surely bijective on the support of Φ

Unimodular Mecke Theorem [BHK18]
Let f be a vertex-shift and $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular network. Then θ_f preserves the distribution of $[\boldsymbol{G}, \boldsymbol{o}]$ if and only if $f_{\boldsymbol{G}}$ is almost surely bijective on $\mathbf{V}(\mathbf{G})$

Vertex/Point-Shift Graph

f-**Graph** of (point/vertex)-shift f:
directed graph with vertices $V(\mathbf{G})$ and edges $\{(v, f(v))\}_{v \in V(\mathbf{G})}$



Euclidean instance: **union of all orbits**, starting from all v

Can be a tree or a forest with compnents $\mathcal{C}^f$

Foliation of a Point/Vertex-Shift

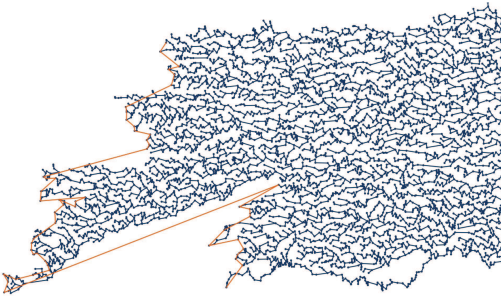
Discrete analogue of the **stable manifold of a smooth dynamics**
Foil partition of the set of points equivalence relation

$$x \sim_f y \Leftrightarrow \exists n \in \mathbb{N}; f^n(x) = f^n(y)$$

f-**foliation**: $\mathcal{L}^f$, equivalence classes of the set of nodes w.r.t. $\sim_f$
The partition $\mathcal{L}^f$ is a **refinement** of the partition $\mathcal{C}^f$
The foil of the root is a **unimodular discrete space**

Illustration: f-graph and foliation of strip PS on a P.P.P.

Φ Poisson P.P. in $\mathbb{R}^2$ Strip Point-Shift The f-Graph has a.s. one component



Foil of origin

Outline

Phase Classification

Theorem [BHK18]
Let f be a covariant point-shift on a unimodular random discrete space $[\mathbf{D}, \mathbf{o}]$.
Almost surely the component C of the origin is a unimodular discrete space that a.s. belongs to **one of the following three phases**:

1. $\mathcal{F}/\mathcal{F}$-**Phase**: C is finite, each of its f-foils is finite
2. $\mathcal{I}/\mathcal{F}$-**Phase**: C is a two-end directed tree with all its f-foils finite
3. $\mathcal{I}/\mathcal{I}$-**Phase**: C is a one-end directed and all its f-foils are infinite

Proof based on the Unimodular Poincaré Recurrence Lemma

F/F Phase

Class $\mathcal{F}/\mathcal{F}$:

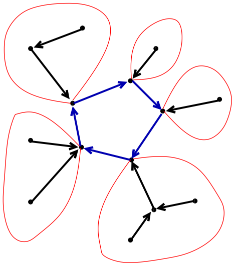
- C is finite (no infinite end)
- each of its f-foils is finite
- $\#$ foils finite

C has a **unique cycle** of length n

Vertices of this cycle: $f^{\infty}(C)$

Example

nearest neighbor point-shift on the P.P.P.



$\mathcal{I}/\mathcal{F}$ Phase

Class $\mathcal{I}/\mathcal{F}$:

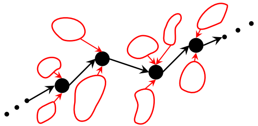
- C is infinite
- Each of its f-foils is finite

C is a **unimodular directed tree**

Each foil has a **junior foil**

$f^{\infty}(C)$: unique **2 end path**

Example: later in the talk



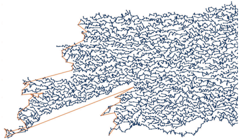
Infinite number of descendants
Finite foil

$\mathcal{I}/\mathcal{I}$ Phase

Class $\mathcal{I}/\mathcal{I}$:

- C is infinite
- All its f-foils are infinite
- Foils order like $\mathbb{N}$ or like $\mathbb{Z}$

C is a **one-ended unimodular tree**



$f^{\infty}(C) = \emptyset$

Examples:

- **Strip PS** on 2 dim. P.P.P. i
- **Canopy tree**

Finite number of descendants
Infinite foil

Outline

Family Trees

Family Tree (FT):
Directed tree T in which the out-degree of each vertex is at most 1

Eternal Family Tree (EFT)
When the out-degrees of all vertices are exactly 1

Rooted FT or EFT:
as above

Parent:
For a vertex v with one outgoing edge vw, $F(v) := w$

Descendants:

- **of generation n** of x: $D_n(x) := \{y : F^{(n)}(y) = x\}$, $d_n(x) := \#D_n(x)$
- Tree of **descendants** $D(x)$ of x, the subtree with vertices $\cup_{n=0}^{\infty} D_n(x)$

Random Family Tree

Random Family Tree:
a random network with values in $\mathcal{T}_*$ almost surely

Unimodular FT:
defined as above via mtp

Proper random FT:
a random $\texttt{FT}$ in which $0 < \mathbb{E}[d_n(\boldsymbol{o})] < \infty$ for all $n \geq 0$

Properties of Unimodular EFTs

Proposition

Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular $\mathtt{FT}$

(i) If $\boldsymbol{T}$ has infinitely many vertices a.s., then it is eternal a.s. Moreover, $[\boldsymbol{T}, \boldsymbol{o}]$ is a proper random $\mathtt{EFT}$, with
- $\mathbb{E}[d_n(\boldsymbol{o})] = 1$ for all $n \geq 0$
- $\mathbb{E}[d(\boldsymbol{o})] = \infty$

(ii) If $\boldsymbol{T}$ is finite with positive probability, then $\mathbb{E}[d_n(\boldsymbol{o})] < 1$ for all $n > 0$

The subtree of descendants of the root of an EFT can be seen as some **generalized branching process**

- No independence assumption

A unimodular EFT is always **critical** in the sense that the mean number of children of the root is 1

Unimodular EFTs and Dynamics on Unimodular Discrete Spaces

From vertex-shift on unimodular network to unimodular EFT

- $[\boldsymbol{G}, \boldsymbol{o}]$: a unimodular network and f a vertex-shift
- $C_{(\boldsymbol{G},\boldsymbol{o})}$: the connected component of the f-graph $\boldsymbol{G}^f$ containing $\boldsymbol{o}$
- Then $[C_{(\boldsymbol{G},\boldsymbol{o})}, \boldsymbol{o}]$, conditioned on being infinite, is a **unimodular EFT**

Conversely

- $[\boldsymbol{T}, \boldsymbol{o}]$: a unimodular EFT
- F: the parent vertex-shift is covariant
- The F-graph is $[\boldsymbol{T}, \boldsymbol{o}]$ itself

Joining of a sequence of directed trees

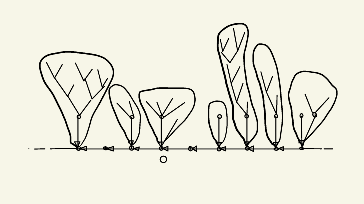
$([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$ a stationary sequence of random rooted trees

Regard each $[\boldsymbol{T}_i, \boldsymbol{o}_i]$ as a Family Tree by directing edges towards $\boldsymbol{o}_i$

Add a directed edge $\boldsymbol{o}_i\boldsymbol{o}_{i-1}$ for each $i \in \mathbb{Z}$

Let $\boldsymbol{o} := \boldsymbol{o}_0$

The resulting random rooted EFT, denoted by $[\boldsymbol{T}, \boldsymbol{o}]$, is the **joining** of the sequence $([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$



Decomposition Result on the $\mathcal{I}/\mathcal{F}$ Phase

If $\mathbb{E}[\#V(\boldsymbol{T}_0)] < \infty$, one can move the root of $\boldsymbol{T}$ to a *typical vertex of* $\boldsymbol{T}_0$:

$$\mathcal{P}'[A] := \frac{1}{\mathbb{E}[\#V(\boldsymbol{T}_0)]}\mathbb{E}\left[\sum_{v\in V(\boldsymbol{T}_0)} 1_A([\boldsymbol{T}, v])\right] \quad \text{probability measure}$$

Theorem [BHK18]
Let $[\boldsymbol{T}, \boldsymbol{o}]$ be the joining of a stationary sequence of trees $([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$ such that $\mathbb{E}[\#V(\boldsymbol{T}_0)] < \infty$.
Let $[\boldsymbol{T}', \boldsymbol{o}']$ be a random rooted EFT with distribution $\mathcal{P}'$

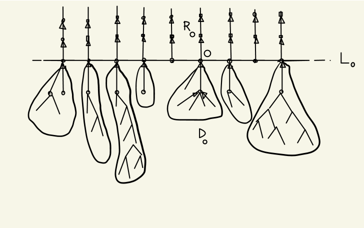
1. $[\boldsymbol{T}', \boldsymbol{o}']$ is a unimodular EFT and of class $\mathcal{I}/\mathcal{F}$ a.s. As a result, all generations of $\boldsymbol{T}$ and $\boldsymbol{T}'$ are finite a.s.
2. Any unimodular non-ordered EFT of class $\mathcal{I}/\mathcal{F}$ can be constructed by joining a stationary sequence of trees

Decomposition Result on the $\mathcal{I}/\mathcal{I}$ Phase

Descendent subtree of the root: D_0 (heavy tailed cardinality)

Ray from the root to the end: R_0

In the **amenable case**, the foil of 0 can be equipped by a bijective point shift b whose orbit is L_0



Theorem [BHK18]
The sequence $\{R_{b^{(i)}(0)}, D_{b^{(i)}(0)}\}_{i \in \mathbb{Z}}$ is **stationary**

Outline

EFTs Everywhere

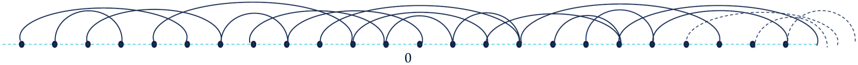
- Renewal EFTs (with O. Sodre & S. Khaniha)
- Record EFTs (with B. Roy Choudhury)
- Evolutionary Trees (Ongoing work with O. Gascuel)

The Renewal EFT

- Graph: grid on $\mathbb{Z}$
- Marks $m(i)$, $i \in \mathbb{Z}$, i.i.d. with distrib. π on $\mathbb{N}^*$
- Unimodular Network

Vertex Shift

$$F(i) = i + m(i)$$



Finite Mean Interarrival Times

Theorem [B & Sodre 22]
Assume that π has finite mean and is aperiodic. Then the F-graph is an EFT (the **Renewal EFT**) which

- is unimodular
- is I/F
- has a covariant subset of individuals with infinite progeny

Infinite Mean Interarrival Times

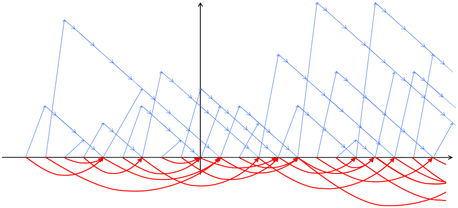
Theorem [B, Khaniha, H 24]

Assume that π has finite mean and is aperiodic. Then the F-graph (**Recurrence Time EFF**)

- Can either be a tree or a forest made of an infinite collection of trees (depending on the tail of the renewal CDF)

In the tree case, the **Renewal EFT**

- is unimodular
- is I/I



Stochastic processes with stationary increments

- Stationary integer-valued sequence $X = (X_n)_{n\in\mathbb{Z}}$ such that their common mean exists
- Stochastic process $S = (S_n)_{n\in\mathbb{Z}}$ is given by

$$S_0 = 0$$

$$n > 0,\ S_n = \sum_{i=0}^{n-1} X_i$$

$$n < 0,\ S_n = \sum_{i=n}^{-1} -X_i$$

- Graph of the process S, $\{(n, S_n) : n \in \mathbb{Z}\}$.

Record graph

- Given a stationary integer-valued sequence $X = (X_n)_{n\in\mathbb{Z}}$, its **record map** $R_X : \mathbb{Z} \to \mathbb{Z}$ is given by

$$i \mapsto R_X(i) = \begin{cases} \inf\{n > i : S_n \geq S_i\} \text{ if inf exists} \\ i \text{ otherwise} \end{cases}$$

  ($S_n \geq S_i$ is equivalent to $\sum_{k=i}^{n-1} X_k \geq 0$).

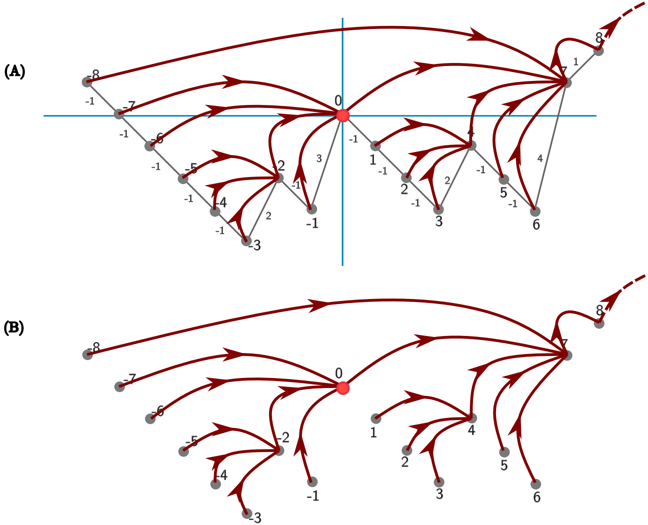
- **The Record Graph** $\mathbb{Z}_X^R$ is the random graph given by

$$\text{vertices: } V(\mathbb{Z}_X^R) = \mathbb{Z}$$

$$\text{Directed Edges: } E(\mathbb{Z}_X^R) = \{(i, R_X(i)) : i \in \mathbb{Z} \text{ and } i \neq R_X(i)\}$$

- $\mathbb{Z}_X^R(i)$ denotes the component of integer i in the record graph

Record graph picture



$S_0 = 0, S_n = \sum_{k=0}^{n-1} X_k$ for $(n > 0)$ and $S_n = \sum_{k=n}^{-1} -X_k$ for $n < 0$

Phase transition of the record graph

Theorem [B & Roy-Choudhury 24]
Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequence of random variables such that their common mean exists. Let $\mathbb{Z}_X^R$ denote the record graph of the network $(\mathbb{Z}, X)$

- If $\mathbb{E}[X_0] < 0$, then a.s. every component of $\mathbb{Z}_X^R$ is of class $\mathcal{F}/\mathcal{F}$.
- If $\mathbb{E}[X_0] > 0$, then a.s. $\mathbb{Z}_X^R$ is connected, and it is of class $\mathcal{I}/\mathcal{F}$ a.s.
- If $\mathbb{E}[X_0] = 0$, then a.s. $\mathbb{Z}_X^R$ is connected, and it is either of class $\mathcal{I}/\mathcal{F}$ or of class $\mathcal{I}/\mathcal{I}$.

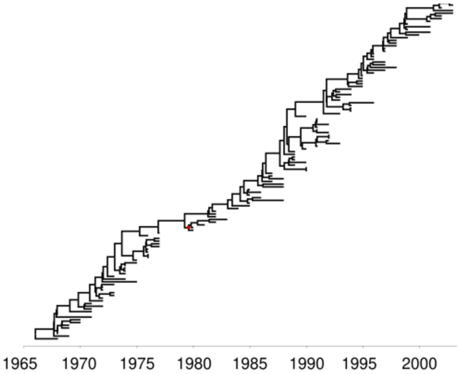
Component of 0 in the record graph is a unimodular tree

Skip-free random walks, $\mathbb{E}[X_0] = 0$ Example

Theorem [F. B., B. Roy Choudhury 24]
Let $X = (X_n)_{n\in\mathbb{Z}}$ be the increments of skip-free to the left random walk and $\mathbb{Z}_X^R$ be the record graph of the network $(\mathbb{Z}, X)$

If $\mathbb{E}[X_0] = 0$, then $[\mathbb{Z}_X^R(0), 0]$, the component of 0 in the record graph is distributed as the ordered $EGWT(\pi)$, where $\pi \stackrel{\mathcal{D}}{=} X_0 + 1$

Evolution: Trigger



The evolution tree of Influenza [Wikipedia]

Evolutionary Trees and their Limits

Reference: Book of M. Steel: Phylogeny, SIAM, 2016

Several classes of models:

- Branching : Bienaymé-Galton-Watson (neutral)
- Coalescent (neutral)
- Yule–Harding (neutral)
- Caterpillar model (non neutral)
- Brunet-Derrida-Mueller-Munier (non neutral)

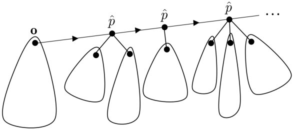
When choosing direction from offspring to parent, and when selecting a node at random as root, each of them admits a local weak limit EFT when letting some size parameter tend to infinity

Ansatz

Prelimit evolution models should belong to one of two phases depending on the phase of their limit

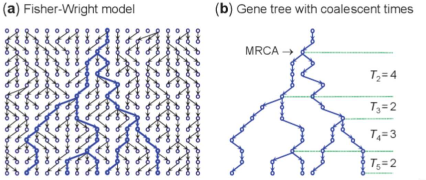
Examples of Limits and Phase Transitions

- Critical branching : **Unimodular Bienaymé-Galton-Watson EFT**
  - $\mathcal{I}/\mathcal{I}$ when variance of offspring distribution is positive
  - $\mathcal{I}/\mathcal{F}$ otherwise



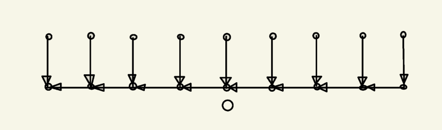
- π distribution on $\{0, 1, 2, 3, \ldots\}$ with mean $m(\pi) = 1$ and $\pi(1) < 1$
- Size-biased distribution of π, $\hat{p}(k) = k\pi(k)$ for all $k \geq 0$

- Coalescent
  - $\mathcal{I}/\mathcal{I}$ when the set of nodes per generation is $\mathbb{Z}$
  - $\mathcal{I}/\mathcal{F}$ otherwise **(hence some neutral models are $\mathcal{I}/\mathcal{F}$)**

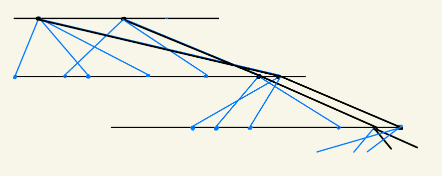


Instance of Coalescent - Local view of case on $\mathbb{Z}$

- **Caterpillar model $\rightarrow$ Caterpillar EFT: always $\mathcal{I}/\mathcal{F}$**



- **Brunet-Derrida-Mueller-Munier model $\rightarrow$ BDMM EFT: always $\mathcal{I}/\mathcal{F}$**
  Individuals reproduce independently like in a branching process
  Each individual has a fitness which is that of its parent plus an increment with positive mean

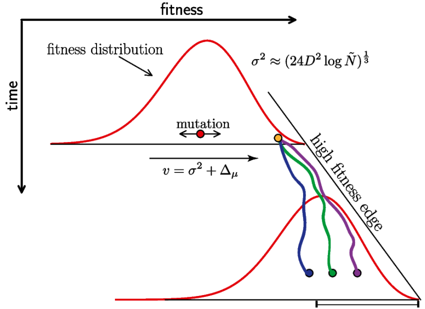


In every generation only the K most fit individuals reproduce

The BDMM Model and the FKPP Universality Class

The Fischer Kolmogorov Petrovskii Piscounov waves

The fitness of the individuals evolves with time as a wave propagating to the right at a constant speed



The BDMM Model belongs to the FKPP Universality Class

The $\mathcal{I}/\mathcal{F}$ phase and the FKPP Universality Class

Let

- $[T, o]$ be an I/F unimodular EFT
- $\{o_l\}_{l\in\mathbb{Z}}$ be the special individual sequence
- $\{[T_l, o_l]\}_{i\in\mathbb{Z}}$ be the trees in the joining decomposition of T
- $g_{l,k}$ be the number of the descendants of order $k \geq 0$ in T_l

Conditionally on $o = o_0$

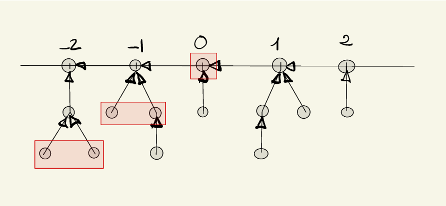
- $\{g_{l,k}\}_{k\geq 0}$ is stationary in l **(joining theorem)**
- $G_0 = \sum_{k\geq 0} g_{0,k}$ has finite mean

$\mathcal{I}/\mathcal{F}$ and FKPP - continued

Define the **fitness** of o_l and of all its descendants in T_l to be l

The fitness of generation l is best represented by the random measure

$$\Phi_l = \delta_l + \sum_{i<0} g_{l+i,l-i}\delta_{l+i}$$



The key observation is that **relative to l,** the random measures Φ_l, have the same probability distributions for all l

Generic extension of the FKPP wave valid for all I/F models

Call **success** of a species (or an individual) the number of its descendant species of all generations.

- In the $\mathcal{I}/\mathcal{I}$ phase, the success of the typical individual is finite but with infinite mean
  A numbering of generations by $\mathbb{Z}$ implies that when **navigating the foil/generation** of the typical species, one finds a **subsequence of individuals with a success tending to infinity** a.s. This sequence of successes is stationary. If it is ergodic, when exploring the foil, one will find species with a success that dwarfs that of any other node visited earlier
- In the $\mathcal{I}/\mathcal{F}$ phase, **success in a generation is infinite for the individual of the generation belonging to the bi-infinite path** (the special individual of this generation) and finite for the others

Outline

Conclusions

Unimodular random discrete spaces

- **Extend Palm calculus** beyond Euclidean
- Lead to several **ergodic theory like results**
- Allow one to define the **dimension** of many classical objects
- Allow one to describe structural properties of **any dynamics on any such space**

Such random structures are **ubiquitous**, with in particular **implications on evolution**

Ongoing research direction:

- **(in)distinguishability** of components and foils.
- For evolution: what makes long term evolution to be of one type or the other?

References

On random unimodular Networks and FTs

- **D. Adous and R. Lyons** *Processes on random unimodular networks*, **EJP**, 2007
- **F.B., M.O. Haji-Mirsadeghi, and A. Khezeli** *Eternal Family Trees and Dynamics on Unimodular Random Graphs*, **Contemporary Mathematics, AMS**, 2018
- **L. Lovász** *Compact Graphings*, **Acta Math. Hungar.**, 161, 2020
- **F.B., M.O. Haji-Mirsadeghi, and A. Khezeli** *Unimodular Hausdorff and Minkowski Dimensions*, **EJP**, 2021

References (continued)

Applications

- **F.B, and A. Sodre**, *Renewal processes, population dynamics, and unimodular trees*, **Journ. Appl. Probab.**, 2019
- **B. Roy-Choudhury** *Records of Processes with Stationary Increments and Unimodular Graphs*, PhD thesis, ENS, ArXiv, 2023
- **F.B, M.O. Haji-Mirsadeghi, and S. Khaniha** *Coupling from the Past for the Null Recurrent MCs*, **Annals Applied Prob.**, 2024
- **F.B, O. Gascuel** *The two classes of large evolution trees* In preparation, 2025

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