ENTANGLEMENT
IN DISORDERED SYSTEMS

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PROGRAM OF THE COURSE

1. Introduction: basic notions of quantum mechanics, bipartite systems, entanglement, reduced density matrix, entanglement quantifiers. A toy model of black hole radiation.
2. Dynamics of two qubits in random environment: general setting, models of environment, basic approximations, random matrix environment (analytical and numerical results).
3. Einstein-Podolsky-Rosen (EPR) paradox and Bell inequalities (dedicated to the Nobel Prize in Physics 2022)
4. Entanglement entropy in extended systems: setting and basic facts for translation invariant systems.
5. Entanglement Entropy of Disordered Fermions: setting, Anderson localization, area law and its violations.

- Reminder
- Entanglement Quantifiers
- Evolution of Two Qubits (Generalities)
- Random Matrix of Evolution of Two Qubits
  - Model
  - Results

Entanglement (reminder)

Entanglement: a complex and delicate (spooky) quantum phenomenon *(Einstein et al 1935, Schrödinger, 1935)* and a widely believed resource of quantum informatics. Q-analog of the statistical dependence in probability theory.

Pure state $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ of a **bipartite** system $\mathcal{S}_A \cup \mathcal{S}_B$ living in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_A$ is **entangled** if $|\Psi_{AB}\rangle \in \mathcal{H}$ is not a **product** (**separable**) state

$$|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle, \ |\Psi_{A,B}\rangle \in \mathcal{H}_{A,B}.$$

Mixed state ρ_{AB} of $\mathcal{S}_A \cup \mathcal{S}_B$ is **entangled** if it is not a convex linear combination of **product** (**separable**, **uncorrelated**) states $\rho_A^j \otimes \rho_B^j, \ j = 1, 2, ..., k$.

Decoherence: various processes of entanglement destruction (by **environment**).

Qubit: $\mathcal{S} = \mathcal{Q}$, a basic entity of quantum information theory (two-level atoms, spins, etc.), $\mathcal{H} = \mathbb{C}^2$.

Entanglement Quantifiers

Quantifier: a non-negative functional of state equals zero on product states and positive on entangled states, ideally faithful, i.e., is zero iff the system is not entangled. Mostly for bipartite systems $\mathcal{S}_A \cup \mathcal{S}_B$.

(i) **Entanglement Entropy** of the RDM $\rho_A = -\operatorname{Tr}_B \rho_{AB}$

$$S[\rho_A] = -\operatorname{Tr} \rho_A \log_2 \rho_{AB} \in [0, \log_2 \dim \mathcal{H}_A].$$

(ii) **Negativity**: $N[\rho_{AB}] \in [0,1]$. Given $\mathcal{S}_A \cup \mathcal{S}_B$ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ of dimensions d_A and d_B, consider its density matrix

$$\rho_{AB} = \{\rho_{j_A j_B, k_A k_B}\}_{j_A,k_A=1, j_B,k_B=1}^{d_A,d_B},$$

i.e., a p.d.o. of trace 1 in $\mathcal{H}_A \otimes \mathcal{H}_B$. Its **partial transpose** is

$$\rho_{AB}^{PT} := \{\rho_{j_A k_B, k_A j_B}\}_{j_A,k_A=1, j_B,k_B=1}^{d_A,d_B}$$

and $\operatorname{Tr} \rho_{AB}^{PT} = 1$ but is not necessarily a p.d.o.!

We have the implication:

$$\rho_{AB} = \rho_A \otimes \rho_B \ \text{ is not entangled} \Rightarrow \rho_{AB}^{PT} = \rho_A \otimes \rho_B^T \text{ is also a p.d.o.},$$

hence, a density matrix. By negation

$$\rho_{AB}^{PT} \text{ is not a p.d.o.} \Rightarrow \rho_{AB} \ \text{ is entangled.}$$

If $\{\lambda_\alpha^{PT}\}$ are the eigenvalues of ρ_{AB}^{PT}, then the **negativity** is

$$N[\rho_{AB}] := -\sum_{\lambda_\alpha<0} \lambda_\alpha^{PT} = 2^{-1}\sum_\alpha \left(|\lambda_\alpha^{PT}| - \lambda_\alpha^{PT}\right).$$

It is algebraic and invariant with respect to local (in S_A or S_B only) unitary transformations, but faithful only for 2×2 and 2×3 bipartite systems.

(iii) **Concurrence**: $C[\rho_{AB}] \in [0,1]$. Its definition is rather involved and is related to another entanglement quantifier, known as **Entanglement (of Formation)** $E[\rho]$ (EE). For a pure state $|\Psi_{AB}\rangle$ (or $|\Psi_{AB}\rangle\langle\Psi_{AB}|$) the EE is just its entanglement entropy

$$E[|\Psi_{AB}\rangle\langle\Psi_{AB}|] := S[\rho_a] = -\operatorname{Tr}_a \rho_a \log \rho_a, \;\; a = A, B.$$

It is the minimum number of standard entangled (say, Bell's) states to create n copies of $|\Psi_{AB}\rangle$ for large n.

For a mixed state ρ_{AB} the EE is

$$E[\rho_{AB}] = \min_{\text{decomp}} \sum_j p_j E[|\Psi_j\rangle\langle\Psi_j|],$$

where the minimum is over all decompositions (c.f. spectral theorem)

$$\rho_{AB} = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|$$

into pure distinct (but not necessarily orthogonal) states $\{|\Psi_j\rangle\}_j$.

The minimization is used to pick out the "irreducible entanglement" of the mixed state. EE is faithful and locally invariant.

The explicit form of the above minimum is known only for two qubits $\mathcal{S}_a = \mathcal{Q}_a,\ a = A, B$:

$$\begin{aligned} E[\rho_{AB}] &= h\left(\left(1 - \sqrt{1 - C^2[\rho_{AB}]}\right) \Big/ 2\right), \\ h(x) &= -x \log_2 x - (1 - x) \log_2 (1 - x), \\ C[\rho] &= \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \end{aligned}$$

$\{\lambda_\alpha\}_{\alpha=1}^4$ are the eigenvalues of $R = \left(\rho_{AB}^{1/2} \tilde{\rho}_{AB} \rho_{AB}^{1/2}\right)^{1/2}$ arranged in non-increasing order,

$$\tilde{\rho}_{AB} = (\sigma_2 \otimes \sigma_2) \rho_{AB}^* (\sigma_2 \otimes \sigma_2)$$

is the so-called spin-flipped state of ρ_{AB}, where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is a Pauli matrix, ρ^*_{AB} is the complex (not hermitian) conjugate of ρ_{AB}.

There are explicit and convenient formulas for $C[\rho]$ directly via the entries of ρ_{AB} for certain classes of two-qubit states.

Nowadays, the concurrence is often used as a measure of entanglement in its own right.

(iv) **Quantum Discord**: $D[\rho_{AB}] \in [0, 1]$. Quantifies all non-classical correlations ("quantumness"), not necessarily entanglement, hence not necessarily zero for product states and manifests the existence of Q-correlations different from entanglement, e.g., just due to the noncommutativity of quantum observables.

For pure states, the quantum discord equals the entanglement entropy.

Entanglement Evolution (Generalities)

Consider a Composite System = Quantum System $\mathcal{S}$ ("small", e.g. several qubits) $\cup$ Environment $\mathcal{E}$ ("big", but not necessarily a macroscopic systems). Its Hamiltonian is

$$H_{\mathcal{S}\cup\mathcal{E}} = H_{\mathcal{S}} \times \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\mathcal{S}} \times H_{\mathcal{E}} + H_{\mathcal{S}\mathcal{E}},$$

its density matrix satisfies ($\hbar = 1$)

$$\frac{d\rho_{\mathcal{S}\cup\mathcal{E}}(t)}{dt} = -i[H_{\mathcal{S}\cup\mathcal{E}}, \rho_{\mathcal{S}\cup\mathcal{E}}],$$

hence

$$\rho_{\mathcal{S}\cup\mathcal{E}}(t) = e^{-itH_{\mathcal{S}\cup\mathcal{E}}} \rho_{\mathcal{S}\cup\mathcal{E}}(0) e^{itH_{\mathcal{S}\cup\mathcal{E}}}, \ t \geq 0,$$

and the **reduced density matrix** is $\rho_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{E}}\ \rho_{\mathcal{S}\cup\mathcal{E}}(t)$.

Goal: study its evolution and that of entanglement quantifiers.

More broadly:

- statistical mechanics (since the XIX century) $10^{23} \sim |\mathcal{S}| \ll |\mathcal{E}|$;

- meso- and nanoscopics, (since the 1980s) $10^1 - 10^3 \sim |\mathcal{S}| \ll |\mathcal{E}|$;

- quantum optics (since the 1960s), $1 \sim |\mathcal{S}| \ll |\mathcal{E}|$;

- quantum information science (since the 1980s), $1 \lesssim |\mathcal{S}| \ll |\mathcal{E}|$).

Here $\mathcal{S}_{AB} = \mathcal{Q}_A \cup \mathcal{Q}_B$,

$$H_{\mathcal{S}_{AB}} = s_A \sigma_3^A \times \mathbf{1}_{Q_B} + \mathbf{1}_{Q_A} \times s_B \sigma_3^B$$

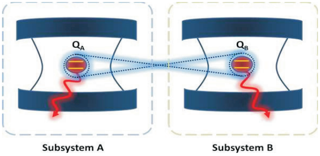
is its Hamiltonian with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

being the Pauli matrices.

There are various schemes for $H_{\mathcal{E}}$ and $H_{\mathcal{S}\mathcal{E}}$. Most widely used are:

TWO QUBITS: DYNAMICS



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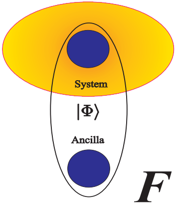
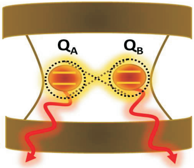


F



C

(i) Scheme I (*two independent* qubits), total Hamiltonian $H_{\mathcal{S}\cup\mathcal{E}} := H_I$

$$\begin{aligned} H_{\mathcal{E}} &= H_{\mathcal{E}_A} \otimes \mathbf{1}_{\mathcal{E}_B} + \mathbf{1}_{\mathcal{E}_A} \otimes H_{\mathcal{E}_B}, \\ H_{\mathcal{S}\mathcal{E}} &= H_{Q_A\mathcal{E}_A} \otimes \mathbf{1}_{Q_B\cup\mathcal{E}_B} + \mathbf{1}_{Q_A\cup\mathcal{E}_A} \otimes H_{Q_B\mathcal{E}_B}, \\ H_{Q_a\mathcal{E}_a} &= v_a\sigma_1^a \otimes J_{\mathcal{E}_a},\ a = A, B \\ H_{\mathcal{S}\cup\mathcal{E}} &= H_{\mathcal{S}_A\cup\mathcal{E}_A} \otimes \mathbf{1}_{\mathcal{S}_A\cup\mathcal{E}_A} + \mathbf{1}_{\mathcal{S}_A\cup\mathcal{E}_A} \otimes H_{\mathcal{S}_A\cup\mathcal{E}_A} \end{aligned}$$

(ii) Scheme F (*free* qubit A (ancilla)), total Hamiltonian $H_{\mathcal{S}\cup\mathcal{E}} := H_F$

$$H_{\mathcal{E}} = \mathbf{1}_{\mathcal{E}_A} \otimes H_{\mathcal{E}_B}, \quad H_{\mathcal{S}\mathcal{E}} = \mathbf{1}_{Q_A\cup\mathcal{E}_A} \otimes H_{Q_B\mathcal{E}_B},$$

(iii) Scheme C (*common* environment), total Hamiltonian $H_{\mathcal{S}\cup\mathcal{E}} := H_C$.

$$\begin{aligned} H_{\mathcal{E}} = \mathbf{1}_{\mathcal{S}} \otimes H_{\mathcal{E}},\ H_{\mathcal{S}\mathcal{E}} &= H_{Q_A\mathcal{E}} \otimes \mathbf{1}_{Q_B} + \mathbf{1}_{Q_A} \otimes H_{Q_B\mathcal{E}} \\ &= (v_A\sigma_A^x \otimes \mathbf{1}_{Q_B} + \mathbf{1}_{Q_A} \times v_B\sigma_B^x) \otimes J_{\mathcal{E}}, \end{aligned}$$

H_I and H_F are "trivial" (product dynamics), but "probe" the entanglement for $t > 0$ induced by the initial conditions.

The whole composite:

$$\rho_{\mathcal{S}\cup\mathcal{E}}(0) = \rho(0) \otimes P_{\mathcal{E}},\ P_{\mathcal{E}} = |\Psi_{\mathcal{E}}\rangle \langle\Psi_{\mathcal{E}}| \,.$$

The qubits, i.e., $\rho(0)$:

Condition 0. The product state: $\rho_0(0) := \rho_{Q_A} \times \rho_{Q_B}$,

$$\rho_{Q_a} = \alpha_a^2 \, |0\rangle_a \langle 0|_a + (1 - \alpha_a^2) \, |1\rangle_a \langle 1|_a \,,\ a = A, B.$$

Condition 1. The pure state: $\rho_1(0) := |\Psi_1\rangle \langle\Psi_1| \,,$

$$|\Psi_1\rangle = \alpha_1 \, |01\rangle + \beta_1 \, |10\rangle \,,\ \alpha_1^2 + |\beta_1|^2 = 1,$$

a *Bell-like* state, a genuine (maximally entangled) *Bell* states are for $\alpha_1 = \beta_1 = \pm 1/\sqrt{2}$.

Condition 2. The pure state : $\rho_2(0) = |\Psi_2\rangle\langle\Psi_2|$,

$$|\Psi_2\rangle = \alpha_2|00\rangle + \beta_2|11\rangle, \ \alpha_2^2 + |\beta_2|^2 = 1$$

another Bell-like state.

Condition 3(k), $k = 1, 2$. The mixed state

$$\rho_{3(k)}(0) = \alpha_3|\Psi_k\rangle\langle\Psi_k| + \frac{1-\alpha_3}{4}\mathbf{1}_4, \ k = 1, 2, \ -1/3 \le \alpha_3 \le 1.$$

known as the *generalized Werner* state, the genuine *Werner* state for $\alpha_k = \beta_k = 1/\sqrt{2}, \ k = 1, 2$. It is a state (a positive definite matrix) for $\alpha_3 \ge -1/3$ and is entangled for $1/3 < \alpha_3 \le 1$ by concurrence.

The states $\rho_0(0)$ are not entangled for all $\alpha_A, \alpha_B \in [0, 1], \ \rho_1(0)$ and $\rho_2(0)$ are not entangled for $\alpha_1 = \alpha_2 = 0, 1$ and $\rho_{3(k)}$ are not entangled for $|\alpha_3| < 1/3$.

It is important that there is a basis in the state space of two qubits where all the above initial condition have the block form

$$\begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{pmatrix}, \quad \rho_{21} = \rho_{12}^*, \ \rho_{43} = \rho_{34}^*,$$

which arises in a number of physical situations and is maintained during the dynamics determined by above Hamiltonians.

Moreover, in this basis we have: for the negativity

$$N[\rho] = -\min\{0, n_1\} - \min\{0, n_2\},$$

$$n_1 = \left(\rho_{11} + \rho_{22} \pm \sqrt{(\rho_{11} - \rho_{22})^2 + 4|\rho_{34}|^2}\right)/2,$$

$$n_2 = \left(\rho_{33} + \rho_{44} \pm \sqrt{(\rho_{33} - \rho_{44})^2 + 4|\rho_{12}|^2}\right)/2$$

and for concurrence

$$C[\rho] = 2\ \max\{0, c_1, c_2\},$$

$$c_1 = |\rho_{34}| - \sqrt{\rho_{11}\rho_{22}},\ c_2 = |\rho_{12}| - \sqrt{\rho_{33}\rho_{44}}.$$

Spin-boson Hamiltonian

Here $\mathcal{S} = \mathcal{Q}$ (a qubit or a two-level system) and

$$H_{\mathcal{Q}} = s\sigma_3, \; H_{\mathcal{E}} = \sum_{k=1}^{n} \omega_k b_k^+ b_k, \; [b_{k'}, b_{k''}] = \delta_{k'k''},$$

$$H_{\mathcal{Q}\mathcal{E}} = \sigma_1 \otimes \sum_{k=1}^{n} c_k (b_k^+ + b_k).$$

Well known in statistical mechanics (relaxation, equilibration, open systems) and quantum optics (atom-light interaction, cavity QED).

The RDM is determined by

$$\sum_{\omega_k \in \Delta} \frac{c_k^2}{\omega_k} \to \int_{\Delta} J(\omega) d\omega, \text{spectral function}$$

A. Legget et al Rev. Mod. Phys. 69 (1980) 1–85; H.-P. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford, 2007.

Is not explicitly solvable in general. One uses

(i) partial solutions (1 or 2 excitation ansatz);

(ii) a variety of approximations, notably the Bogolyubov-van Hove (Born-Markov) asymptotic regime

$$t \to \infty, \ v \to 0, \ v^2 t \to \tau \in [0, \infty) \text{ is slow time.}$$

It was found in the 2000s by using (i) or (ii) for the spin-boson Hamiltonian and its two-qubit analogs:

(a) the vanishing of entanglement at finite moments of time (Entanglement Sudden Death, ESD)

(b) the revivals of entanglement at finite moments of time (Entanglement Sudden Birth, ESB).

T. Yu, J.H.Eberly, Science 323 (2009) 581-601; Lo Franco R. et al, Int. J. Mod.Phys.B 27 (2013) 1345053.

Random Matrix Hamiltonian

One Qubit

Replace boson operators in the spin-boson Hamiltonian by random matrices:

$$H_n = s\sigma_3 \otimes \mathbf{1}_n + \mathbf{1}_2 \otimes M_n + v\sigma_1 \otimes W_n,$$

where M_n is a $n \times n$ random hermitian matrix with the density of states (DOS)

$$\nu_0^{(n)}(E) = n^{-1} \sum_{j=1}^{n} \delta(E - E_j) \to \nu_0(E),$$

and a smooth $\nu_0(E)$ (playing the role of $J(\omega)$ in the spin-boson Hamiltonian), W_n is the $n \times n$ random hermitian matrix with independent (modulo symmetry conditions) entries such that

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{W_{jk}^2\} = 0, \quad \mathbf{E}\{|W_{jk}^2|\} = (1 + \delta_{jk})/n,$$

e.g. GUE (Gaussian Unitary Ensemble), cf. the macroscopic universality of random matrix theory.

RMT environment: noisy or typical. Rather one body (mean field or multiconnected, e.g. graphs, quantum dots, human brain etc.).

Motivations: random matrices models "typical" operators, hence "typical" reservoirs and interactions.

It is assumed again for the initial density matrix of compound:

$$\rho(0) \otimes P_k,$$

where $\rho(0)$ is a 2×2 positive definite matrix of unit trace and P_k is the orthogonal projection on the (non-degenerate) state of energy E_k of the reservoir (recall microcanonical ensemble in statistical mechanics, standard in quantum information science).

The reduced density matrix $\rho(t)$ of the qubit is now random, thus, one studies for large n, $E_k \to E$:

- its mean as $n \to \infty$.
- its selfaveraging property (representativity of means), e.g. the vanishing of variance of ρ as $n \to \infty$,

It can be shown that

(i)

$$\mathbf{Var}\{\rho_{\alpha,\beta}(E_k,t)\} \le C(tv)^2/n, \ \alpha,\beta = 0,1,$$

i.e., that $\rho(E_k,t)$ is selfaveraging for $1 \lesssim tv \ll n^{1/2}$;

(ii) the model is asymptotically exactly solvable in the limit $n \to \infty$ assuming that $\nu_0(E) > 0$, in particular:

- the limiting reduced density matrix is a solution of certain self-consistent equations,
- its evolution is not local in time (non-Markovian), i.e., $\rho(t) \neq e^{-tL}\rho(0)$, not Lindblad (quantum Fokker-Planck),
- $\rho(E,\infty)$ depends in general on $\rho(0)$, i.e., the random matrix reservoir is not "big" enough to equilibrate ("thermalize") $\mathcal{S}$, a typical situation for non-macroscopic (meso- and nano-)systems.

The double limit $v \to 0, \quad t \to \infty, \quad \tau = tv^2$, done after the limit $n \to \infty$:

$$\begin{aligned}\rho_{\alpha,\alpha}(\tau) &= \frac{\nu_0}{\nu_0+\nu_+}\rho_{\alpha,\alpha}(0) + \frac{\nu_0}{\nu_0+\nu_-}\rho_{-\alpha,-\alpha}(0) \\ &+ \frac{\nu_+}{\nu_0+\nu_+}\rho_{\alpha,\alpha}(0)e^{-\tau\Gamma_\alpha} - \frac{\nu_-}{\nu_0+\nu_-}\rho_{-\alpha,-\alpha}(0)e^{-\tau\Gamma_{-\alpha}},\end{aligned}$$

where $\nu_\alpha := \nu_0(E+2s\alpha),\ \alpha = 0, \pm$ and ν_0 is the DOS of the environment, $\Gamma_\alpha = 2\pi\,(\nu_0+\nu_\alpha)$, i.e., is a version of the *Fermi Golden Rule*

Usual: exponential decay to the mixed $\rho(\infty)$ even if $\rho(0)$ is pure according the Fermi Golden Rule.

Unusual: (i) $\rho(\infty)$ is not an equilibrium (Gibbs) state, moreover, depends on $\rho(0)$ unless $\nu_{0+}\nu_{0-} = \nu_0^2$;

(ii) $\rho_{\alpha,\alpha}(\tau)$ is a Markov process iff $\nu_0 = \nu_+ = \nu_-$, i.e., the "flat" DOS.

Random Matrix Hamiltonian

Two Qubits

Recent interest for non-Markov evolution encountered in many experiments motivates the use of this dynamics in the context of quantum information science ("small" system consisting of 2 qubits).

Thus, consider a the two-qubit random matrix analog of the above one-qubit evolution.

This, together with the above initial Conditions 0 - 3, yields the models (dynamics + initial conditions) $Nn, \ N = I, F, C, \ n = 0, 1, 2, 3$ of the two-qubit evolution in the disordered environment described by random matrices.

(i) *Hamiltonian* H_I (independent environments)

$$\begin{aligned} H_I &= H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\mathcal{S}} \otimes (M_n \otimes \mathbf{1}_{\mathcal{E}_B} + \mathbf{1}_{\mathcal{E}_A} \otimes M_n) \\ &+ v\sigma_1^A \otimes \mathbf{1}_{\mathcal{S}_B} \otimes W_n^{\mathcal{E}_A} \otimes \mathbf{1}_{\mathcal{E}_B} + v\mathbf{1}_{\mathcal{S}_A} \otimes \sigma_1^B \otimes \mathbf{1}_{\mathcal{E}_A} \otimes W_m^{\mathcal{E}_B}. \end{aligned}$$

with independent $W_n^{\mathcal{E}_a}$, $a = A, B$ (independent of M_n if it is random).

(ii) *Hamiltonian* H_F ((A-qubit is free))

$$H_F = H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\mathcal{S}} \otimes M_n \otimes \mathbf{1}_{\mathcal{E}_A} + v\sigma_1^B \otimes \mathbf{1}_{\mathcal{S}_a} \otimes W_n^{\mathcal{E}_B} \otimes \mathbf{1}_{\mathcal{E}_A}$$

(iii) *Hamiltonian* H_C (common environment)

$$H_C = H_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\mathcal{S}} \otimes M_n + v(\sigma_1^A \otimes \mathbf{1}_{\mathcal{S}_B} + \mathbf{1}_{\mathcal{S}_A} \otimes \sigma_1^B) \otimes W_n^{\mathcal{E}},$$

Recall that for two qubits

$$H_{\mathcal{S}} = s_A\sigma_3^A \times \mathbf{1}_{\mathcal{S}_B} + \mathbf{1}_{\mathcal{S}_A} \times \sigma_3^B.$$

The dynamics I (independent) and F (free) are easily constructed out of the above one-qubit dynamics, while the dynamics C (common) has to be constructed anew.

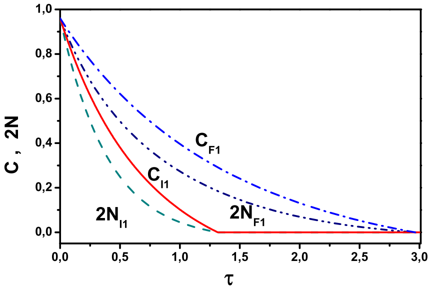
In the models In and Fn the qubits are connected only via initial conditions and the results depend only on $a = \nu_0(E + 2s)/\nu_0(E))$ and $b = \nu_0(E - 2s)/\nu_0(E)$.

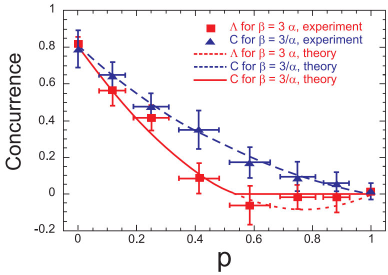
In the models Cn the qubits are connected both via initial conditions and the common reservoir (indirect qubit-qubit interaction) and the results depend on a number of parameters. Thus, we use the Lorentzian DOS:

$$\nu_0 = \gamma/\pi(E^2 + \gamma^2).$$

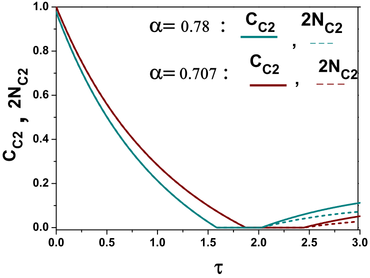
Here is a collection of results obtained by combining analytical and numerical tools.

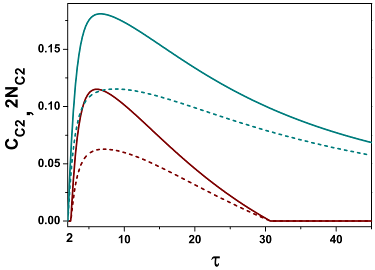
Results



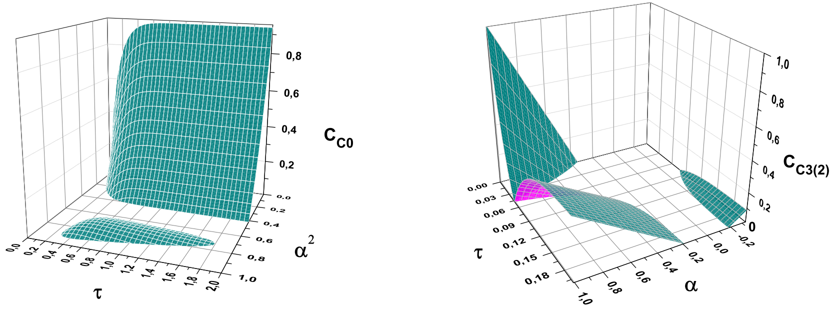


Left: Concurrence and negativity for models $F1$ and $I1$ (pure Bell-like initial states, $\alpha_1 = 0.6$) with parameters: $a = 1.2,\ b = 0.05$. Observe the ESD phenomenon, simultaneous in C and $2N$. Right: M.P.Almeida et al, *Experimental Observation of Environment-induced Sudden Death of Entanglement*, Science 316 (2007)579. The qubits are encoded in the polarizations of photons, $p = 1 - e^{-\tau}$.

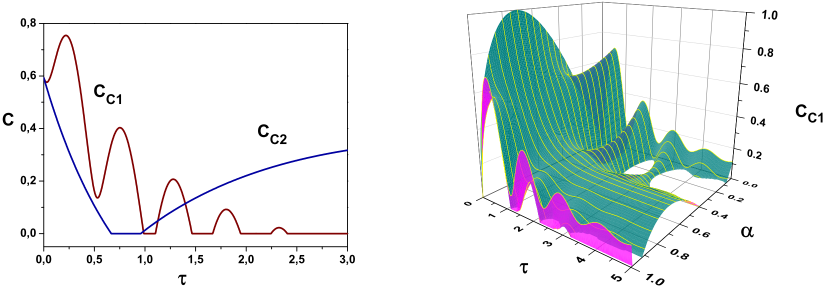




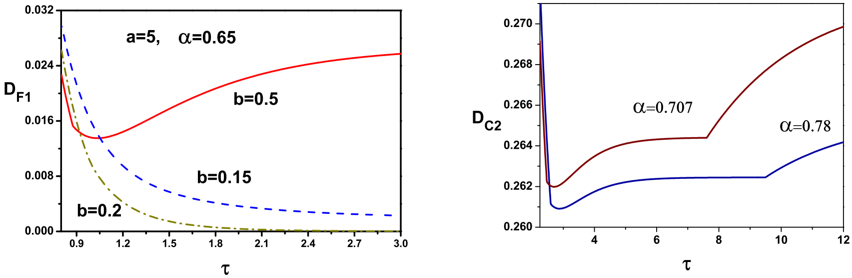
Concurrence and negativity for the model *C*2 (Bell-like pure initial states, $\alpha_2 = 0.78$, $0.70 = 1/\sqrt{2}$). Left: The first pair of branches and the beginning of the second pair of branches (second life!). Observe the ESD and ESB phenomena. Right: The second pair of branches, both are not monotone. The blue pair tends to 0.024 and 0.021 at infinity (entanglement trapping or eternal quantum life?), while the brown pair dies simultaneously at $\tau_{ESD} < \infty$. The horizontal scale on the left is 11 times larger and the vertical scale is 5 times larger than those on the right. Recall the entanglement distillation.



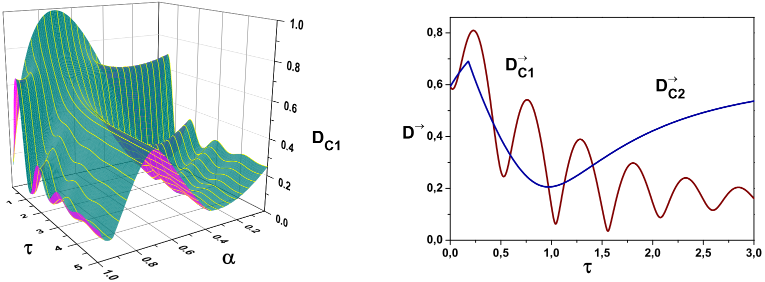
The role of dynamical correlations. Left: Concurrence for the model $C0$ (product initial states, $\min \tau_{ESD} = 0$, $\alpha := \alpha_A = \alpha_B \in (0, 1]$). Small α: fast growth for $\tau > \min \tau_{ESB}$, slow decay for large τ, zero or non-zero at infinity. Adjacent to 1 α: an "island" of non-zero entanglement with $0 < \tau_{ESB} < \tau_{ESD} < \infty$. Right: Concurrence for the model $C_{3(2)}$ (generalized Werner initial states, not entangled for $|\alpha| \leq 1/3$, i.e., with $\min \tau_{ESD} = 0$). Small α: an "island". Intermediate α: no entanglement. Adjacent to 1 α: $0 < \tau^{(1)}_{ESD} < \tau^{(2)}_{ESB} < \tau^{(2)}_{ESD}$ finite or infinite.



Strong dependence on initial conditions. Left: Concurrence for the models $C1$ and $C2$ (Bell-like pure initial states, $\alpha := \alpha_1 = \alpha_2 = 0.95$) with the same parameters $E = 1.38$, $\gamma = 0.04$. $C1$: multiple alternating ESD and ESB with $\min \tau_{ESD} > 0$, $\max \tau_{ESD} < \infty$. $C2$: $0 < \tau_{ESD} < \tau_{ESD} < \infty$. Right: Concurrence for the models $C1$. Displays various behaviors as function of τ for various α_1, monotone decreasing to a non-zero at infinity only for $\alpha_1 = 1/\sqrt{2}$ (maximally entangled initial state).



Left: Quantum discord for the model $F1$ (Bell-like pure state with $\alpha_1 = 0.65$) with various DOS of environment:
$a := \nu_+/\nu_0 = 5,\ b := \nu_-/\nu_0 = 0.5, 0.2, 0.15$. The values at infinity are 0.027, $1.82 \cdot 10^{-3}$ and 0. Decays monotonically to zero at infinity only in the "Markov" point $ab = 1$. May grow at infinity. Right: Quantum discord for the model $C2$ (Bell-like pure initial state). 1) Upper curve - maximally entangled initial condition $\alpha_2 = 1/\sqrt{2} = 0.7$. 2) Lower curve - $\alpha_2 = 0.78$. Both curves show the discord freezing.



Left: Quantum discord for the model $C1$ (Bell-like pure $\alpha_1 = 0.65$ with the same as those for the concurrence. D_{C1} is never zero and is not monotone, except $\alpha = 1/\sqrt{2}$, i.e., the maximally entangled (Bell) initial state). Observe the similarity with the behavior of the concurrence. Right: Trace distance discord for the models $C1$ and $C2$ with $\alpha_1 = \alpha_2 = 0.955$. Quite different behavior although the both are never zero

Random matrix models of qubits embedded into an environment demonstrate a wide variety of properties of the evolution of one and two qubits, including vanishing their entanglement at finite moments, the subsequent entanglement revivals, non-Markovian evolution, etc. Some of these properties have been found before in special versions of the macroscopic and translation invariant spin-boson model and/or by using certain approximations. Our results, obtained for non-macroscopic and disordered environment, demonstrate the robustness (universality) of the above and other essential properties of entanglement evolution. Being combined with other processes of quantum information theory (e.g. entanglement distillation), they can lead to a considerably slower decay of entanglement up to its asymptotic persistence for large time.